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MATHEMATICAL DESCRIPTION OF THE GPS MULTISATELLITE FILTER/SMOOTHER

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This report provides a detailed mathematical description of the GPS Multisatellite Filter/Smoother system of computer programs. Orbit, clock, and additional state and covariance estimates are determined simultaneously for several GPS satellites in this system. The square root information implementation of the Kalman filter/RTS smoother was adopted. Stochastic clock and orbit models are included. Smoothed range, correlated range difference, and interferometric-type derived observations based on simultaneous observations can be processed. This system of programs was designed to be used in production as well as for research into improving the GPS ephemeris and clock estimate accuracies and supporting evaluations. This system is part of a larger system of programs for orbit determination under development called OMNIS. Keywords.

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FOREWORD

The Multisatellite Filter/Smoother system of computer programs was developed by the Space and Surface Systems Division of NSWC in support of the Defense Mapping Agency various geodetic applications of GPS and the Navy's Strategic Systems Program Office/Applied Physics Laboratory SATRACK project. The principal software developers were D. Clark, K. Davis, E. Durling, M. Eward, and H. Ball as members of the Physical Sciences Software Branch. This software is part of the new OMNIS orbit computation system under development. I am grateful to MCSI and, especially, Susan Bowen for helping in the production of this report.

Approved by:

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INTRODUCTION

The Global Positioning System (GPS) is a passive, all-weather, worldwide navigation system that utilizes ultrastable atomic frequency standards to provide navigation messages and signals of the required accuracies. The Space and Surface Systems Division of NSWC has been tasked by the Defense Mapping Agency and the Navy's Strategic Systems Program Office to develop estimation techniques for highly accurate ephemeris and clock determination (after-the-fact) for geodetic and the SATRACK applications, respectively. For geodetic applications, such as point positioning and satellite-to-satellite tracking involving GPS, the user is interested in accurate ephemeris and clock information at all times and with minimal discontinuities. For the SATRACK application (missile tracking primarily during powered flight), the user (Applied Physics Laboratory/Johns Hopkins University) is interested in this information over a particular geographical area for a short time span. The goal of this development effort was to produce common software that has the flexibility to optimize the orbit and clock estimates as required and generates the required products for these two primary applications. In addition, this software was to have extensive diagnostics and be easily reconfigured and/or modified to be used as a research and development tool for accuracy evaluations and other studies.

The result of this development effort is called the GPS Multisatellite Filter/Smoother (MSF/S) system of programs. Smoothed range, correlated range difference, and two interferometric-type derived observations based on simultaneous range observations can be processed. A Kalman filter followed by a smoother was chosen as the estimation technique for several reasons. The unmodeled accelerations acting on the satellites (due to modeling deficiences in the gravity field, radiation pressure, and thermal radiation models, as well as control system induced effects) and the random behavior of atomic clocks are best handled by stochastic estimation techniques, i.e., Kalman filtering. Fixed-interval smoothing can be accomplished because processing is after-the-fact; thus, estimates at a given time can be based on both past and future data. A square root information filter/smoother (SRIF/SRIS) formulation utilizing matrix triangularization techniques was selected for this system primarily because of its numerical accuracy and stability. In addition, this smoothing procedure requires that only inverses of upper triangular matrices be computed, as opposed to inverses of full matrices–as is the case for most covariance-related smoother implementations. Also, if smoothed covariance information is not required, the upper triangular matrix to be inverted is only for a subset of the parameters in the selected implementation. These ideas will be expanded upon in subsequent sections.

The multisatellite capability was adopted because it is the optimum way to separate the satellite clock offsets (of interest to users) and the station clock offsets (of no interest to users). This is because of the simultaneous tracking of a given satellite by two or more stations in conjunction with simultaneous tracking of several satellites by each station. Intersatellite orbit and clock covariances are required for the SATRACK project and are only available with simultaneous processing. Also, this allows the system of programs to accommodate doubly-differenced data types involving two satellites and to eventually be expanded to accommodate the proposed cross-link ranging (satellite-to-satellite tracking) data. In addition, the adoption of a multisatellite capability affected the formulation of the Kalman filter process noise incorporation dealing with minimizing array storage requirements, to maximize the number of satellites that can be processed simultaneously.

The purpose of this report is to provide the assumptions and mathematical details for the GPS MSF/S system of programs. First, the overall GPS data flow is given to introduce the reader to the preliminary computations required before executing the MSF/S. Then, the estimation concepts employed are discussed; followed by the timeline definitions; and a detailed description of the adopted state equations, underlying models, and resulting process noise covariance matrices required by the Filter. Next, the observation equations and the corresponding partial derivatives of the data with respect to the parameter set are given. Then, the overall Filter/Smoother processing flow is provided to establish the groundwork for the detailed descriptions of the filtering and smoothing algorithms, the generation of the solution and diagnostics, and the propagation of trajectories. Derivations and other relevant technical details are supplied in the appendices to enhance understanding the main text.

GPS DATA FLOW

The possible observations are pseudorange and range difference derived from integrated Doppler or phase measurements. Preprocessing of observations may be required for various reasons. Smoothing of pseudorange measurements, accummulation and differencing of integrated Doppler over longer intervals, elimination of duplicate data, and assignment of observation standard deviations and pass numbers may be required. The pass-oriented data must be merged into a time-ordered format, and time-ordered data in one format must be converted into another format. Data from various sources must be merged for the time span of interest. The MSF/S format contains the following information for each observation: time of observation, observation value, standard deviation(s), editing flag, data type, integration interval for range difference data, station number, satellite number, channel or tracker number, pass number, source of weather data, as well as temperature, pressure, and relative humidity in order to make the tropospheric refraction correction. Figure 1 gives a simplified flow of the GPS data after the observations have been preprocessed.

Reference trajectories covering the span of interest are required for both the Corrector/Editor and MSF/S systems of programs. These trajectories are integrated either in GPS or UTC time using initial conditions from a previous fit or obtained from the GPS Operational Control System. For the Corrector/Editor system in its normal mode of operation, the only information required from the trajectories is inertial position and velocity (obtained by numerical differentiation) as a function of time and polar motion. However, for the MSF/S system the reference trajectories and the corresponding Lagrangian interpolation procedures provide the following additional information:

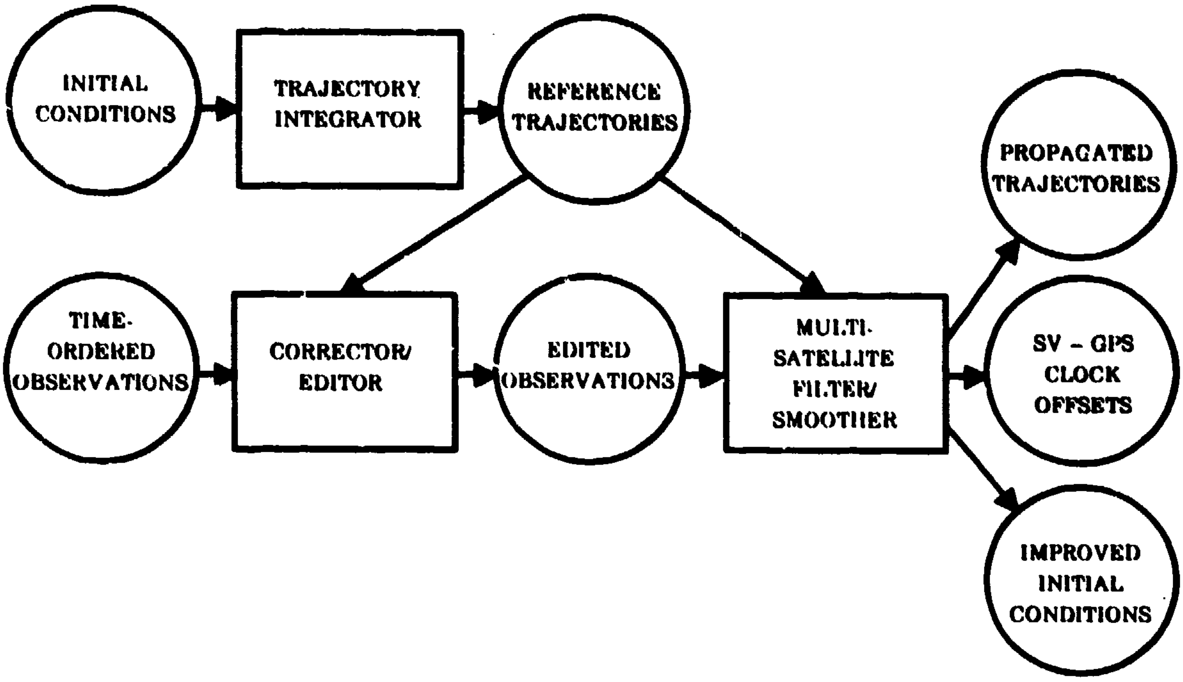


FIGURE 1. GPS DATA FLOW

1. Homogeneous variational equation solution for the epoch orbital elements, i.e., partials of current time position and velocity with respect to epoch orbital elements needed to relate changes at epoch to all other times. (The gravity field model on which the filtering is based is thus determined by the integrator along with the inertial reference frame.)

2. Initial values for the radiation pressure model parameters and the body-axis-to-inertial rotation matrix and body-axis radiation pressure accelerations at the required times. (These quantities are required for the computation of certain partial derivatives associated with solving for stochastic radiation pressure parameters and permit evaluation of the complex radiation pressure force model only during the integration procedure.)

3. Partial derivatives required to solve for nonstochastic radiation pressure parameters, thrusts, polar motion, and gravity field model coefficients.

The main requirement on the reference trajectories is that they are within the linear region of convergence relative to the true trajectory, since a linearized (not extended) Kalman filter has been adopted. However, if a reference trajectory is not within the linear region the Filter can be used in a batch emulation mode to obtain an improved set of initial conditions for reintegration. Reference 1 contains a detailed description of the reference trajectory integration procedures and the associated force models.

The Corrector/Editor system of programs has two purposes. The first purpose is to correct the data to instantaneous geometric range (or range difference), plus clock offsets, measurement noise, and residual unmodeled and random effects. This is accomplished by adjusting the measurements to account for time transmission effects, the effect of the displacement of the antenna system's electrical phase center from the satellite's center of gravity, tropospheric refraction effects, the periodic component of the relativistic effect on the satellite clock, and the solid earth tide effect on the station height. Range and range difference data can be processed by this system of programs with corrections appropriate to each type and source of data made. The second purpose is to edit the data. The Editor does polynomial fits to residuals, formed by differencing a computed value based on the reference trajectory and nominal clock information with the corrected observed value. Consistency checks of the after-fit residuals are used to identify bad points. By-products of this editing procedure are estimates of the nominal clocks for each satellite and station, as well as the time of occurrence and approximate magnitude of any time or frequency jump events that may be observed in the residuals. Reference 2 contains a detailed description of the Corrector/Editor system of programs.

Once the range data has been corrected and edited, another program can be used to search for simultaneous observations and form the derived interferometric data types--differenced range and doubly-differenced range. Differenced range is obtained by differencing ranges from two stations to the same satellite. This eliminates the satellite clock from the observations. Doubly-differenced range is obtained by differencing two differenced ranges involving the same pair of stations but different satellites. This also eliminates the station clocks from the observations.

In addition to the reference trajectories, edited observations (including observation sigmas and station coordinates), and nominal clock information (including events and offsets between the master or reference clock and GPS time), the MSF/S system of programs requires various inputs including: overall program flow and identifying information, quantities defining time spans for the fit and each data type, lists of satellites and stations to be processed, data deletion criteria, minimum observation sigmas for each data type, the parameter set and *a priori* statistics, and information for SATRACK processing if required. Output products of the MSF/S system of programs include: propagated trajectories, satellite clock offsets from GPS time (both time and frequency), improved initial conditions to initiate follow-on processing, plots of corrections and their corresponding sigmas, residual and signal-to-noise plots, correlation coefficient matrices, SATRACK intersatellite and intertime covariance matrices, updated station coordinates, and updated polar motion information.

ESTIMATION CONCEPTS

The GPS MSF/S estimation procedure can be viewed as an adjustment of a model to best fit, in a minimum mean-squared error sense, the available observations which are a function of that model. The possible observations are range, range difference, differenced range, and doubly-differenced range--all involving tracking of the satellites by stations on the surface of the Earth. The model consists of a trajectory model, clock models for both the satellite and station clocks, and various other parameters related to the measurements.

Since the ordinary differential equations describing a satellite trajectory are nonlinear and the Kalman filter equations assume a linear state model (and a linear measurement model), a linearization about a reference trajectory must be performed. In addition, since atomic clocks sometimes exhibit anomalous step changes in time and/or frequency or may be adjusted deliberately, a nominal clock including approximate values for all step changes is also required, so the stochastic clock models do not have to accomodate large jumps. Linearization means that the states of the Kalman filter are actually corrections to the nominal model parameters and that the measurements are processed as residuals. Therefore, partial derivatives are required that relate the states at one time to another time (state transition matrices) and that relate the measurements to the states (measurement matrices). All partial derivatives are evaluated based on the reference trajectory and the nominal parameter values. There is no relinearization of the measurement model relative to the estimated states. The Δ notation is used throughout this report to indicate that corrections to nominal model parameters are actually being estimated.

It is important to realize that a Kalman filter is a combination of a parameter estimation technique and a set of equations that define how the state and its associated covariance at one moment in time, t_j, are related to the same quantities at another time, t_{j+1}. The general form of the stochastic state equations (in discrete terms) is given by:

$$\Delta \mathbf{x}_{j+1} = \phi_j \Delta \mathbf{x}_j + G \mathbf{w}_j \tag{1}$$

where $\Delta \mathbf{x}_j$ = state at t_j

ϕ_j = $\phi(t_{j+1}, t_j)$ = nonsingular transition matrix relating the state at t_j to the state at t_{j+1}

$\mathbf{w}_j$ = vector of white process noise terms with nonsingular covariance matrix Q_j, dim $\mathbf{w} \leq$ dim $\Delta \mathbf{x}$

and G = matrix of ones and zeroes required to make dim $G\mathbf{w}$ = dim $\Delta \mathbf{x}$.

This general form is assumed when describing the processing steps in a standard Kalman filter. The specialized form of the state equations adopted in the MSF/S system, their underlying models, and the corresponding Q matrices are described in the **STATE EQUATIONS** section.

The discrete form of the linear measurement model is given by:

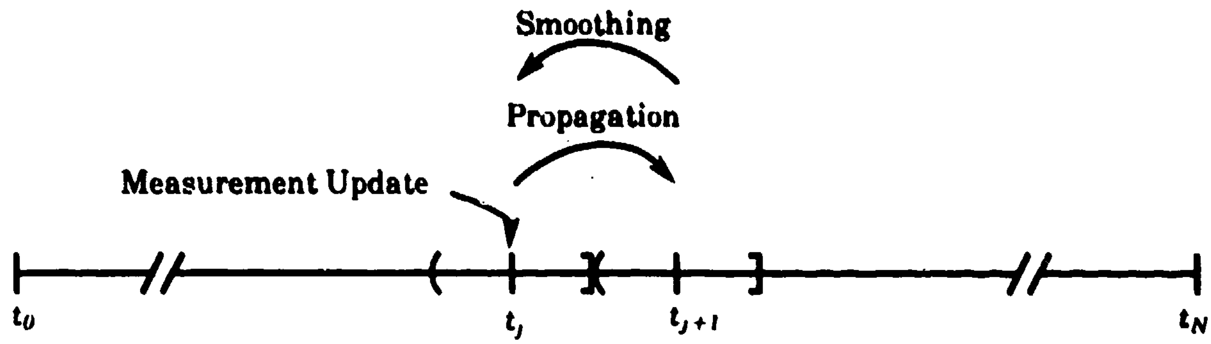
$$\mathbf{z}_j = A_j \Delta \mathbf{x}_j + \mathbf{v}_j \tag{2}$$

where $\mathbf{z}_j$ = measurement vector at t_j

A_j = measurement matrix at t_j

and $\mathbf{v}_j$ = measurement noise vector at t_j.

It is assumed that the observations have been whitened and decorrelated, so the measurement noise covariance matrix is the identity matrix, i.e. $P_v = I$. It is also assumed that the process noise, $\mathbf{w}_j$, and measurement noise, $\mathbf{v}_j$, are uncorrelated and that the state estimate, $\Delta\tilde{\mathbf{x}}_0$, and its associated covariance matrix, $\tilde{P}_0$, are given at t_0.



$\{t_j\}$ = mini-batch times ( ] = mini-batch intervals $\Delta t = t_{j+1} - t_j$

FIGURE 2. SIMPLIFIED TIMELINE DEFINITION

Figure 2 gives a simplified timeline definition. A complete timeline definition is given in the next section. For a standard linear Kalman filter the processing starts by initialization of the state and its associated covariance matrix at t_0, the epoch of the fit span, and steps forward in time. Assume for illustration purposes that observations only occur at the times $\{t_j\}$. In the following equations ~ indicates a predicted quantity, ^ indicates a filtered quantity, and * indicates a smoothed quantity. A measurement update is done at t_j by forming the predicted residuals and their covariance matrix, computing the gain matrix K_j, and using these quantities along with the measurement and measurement matrix to update the estimates of the state and the covariance matrix.

$$\hat{\Delta \mathbf{x}}_j = \tilde{\Delta \mathbf{x}}_j + K_j(\mathbf{z}_j - A_j\tilde{\Delta \mathbf{x}}_j) \tag{3}$$

$$\hat{P}_j = (I - K_jA_j)\tilde{P}_j \tag{4}$$

where

$$K_j = \tilde{P}_j A_j^T(A_j\tilde{P}_j A_j^T + I)^{-1} \tag{5}$$

$\mathbf{z}_j - A_j\tilde{\Delta \mathbf{x}}_j$ is the predicted residual vector and

$A_j\tilde{P}_j A_j^T + I$ is the predicted residual covariance matrix.

The next step is to propagate the state and the covariance matrix to time t_{j+1}. The state propagation uses the state transition matrix ϕ_j and assumes that the process noise term $\mathbf{w}_j$ is zero. The covariance matrix is propagated by doing a deterministic mapping using the ϕ_j matrix and then adding in Q_j, the state or process noise covariance matrix. This Q_j matrix is the only difference between a Kalman filter and a sequentially implemented batch weighted least squares estimator.

$$\tilde{\Delta \mathbf{x}}_{j+1} = \phi_j\hat{\Delta \mathbf{x}}_j \tag{6}$$

$$\tilde{P}_{j+1} = \phi_j\hat{P}_j\phi_j' + GQ_jG' \tag{7}$$

The measurement update/propagate pair of operations is continued until a measurement update has been done at t_N.

Smoothing is then accomplished by recursively computing state and covariance estimates backwards, one step at a time, using the Rauch-Tung-Striebel (RTS) equations. Smoothed state estimates at t_j are the filter estimates, plus a smoother gain matrix multiplied times the difference of the smoothed and predicted states at t_{j+1}. The gain matrix is a function of the filtered covariance estimate at t_j, the predicted covariance estimate at t_{j+1}, and the state transition matrix ϕ_j. The smoothed covariance estimate is the filtered estimate, plus a term that is a function of the difference of smoothed and predicted covariance estimates at t_{j+1} and the gain matrix C_j.

$$\Delta \mathbf{x}_j^* = \hat{\Delta \mathbf{x}}_j + C_j(\Delta \mathbf{x}_{j+1}^* - \tilde{\Delta \mathbf{x}}_{j+1}) \tag{8}$$

$$P_j^* = \hat{P}_j + C_j(P_{j+1}^* - \tilde{P}_{j+1})\ C_j' \tag{9}$$

where

$$C_j = \hat{P}_j \phi_j^T \tilde{P}_{j+1}' \tag{10}$$

This recursive process is continued until t_0 is reached. Reference 3 contains an excellent introduction to linear Kalman filtering and smoothing.

Kalman filters can easily be restarted at any time in the middle of the fit span. Assuming that the filter was previously stopped after propagation from $t_{\ell-1}$ to t_ℓ, restarting the filter simply consists of initializing the state and covariance matrix estimates with their predicted values at t_ℓ and then doing a measurement update at t_ℓ. This procedure has been adopted in the MSF/S system.

In the case of GPS it is necessary to use the mini-batch concept, because not all observations lie exactly at the t_j times and reducing the number of steps required increases the efficiency of the computations. In the mini-batch concept all observations in the interval $\left(t_j - \frac{\Delta t}{2}, t_j + \frac{\Delta t}{2} \right]$ are processed in a batch mode. This assumes that the process noise contribution to state uncertainties can be ignored over much shorter periods than the time constants of the stochastic processes. This means that the state noise covariance matrix $\mathbf{Q}_j$ is only added into the covariance matrix estimate when propagating from t_j to t_{j+1}. This essentially averages out the random effects over the interval chosen, which primarily affects the clock estimates for the intervals used for GPS (≤ 1 hr). Another effect is that solutions are only available at the mini-batch steps. However, that part of the state equations involving orbit-related parameters was chosen so that deterministic propagation of the trajectory corrections between t_j and t_{j+1} is exact.

This approach also allows the mini-batch step to be changed in the middle of the fit span. This technique was adopted for the SATRACK application of the MSF/S system. Since this application requires optimum and dense estimates only during a subspan of the entire fit span, a reduced mini-batch step size span is defined. The concepts given above apply, except Δt is reduced to a smaller value. The transition regions are handled to ensure no observation is processed twice. This technique is also useful for looking in detail at a particular time span to locate precisely when an anomaly occurred.

The square root information implementation of the estimation equations was selected for the MSF/S system. This implementation is mathematically equivalent to the classical Kalman filter/RTS smoother approach. A square root implementation was selected because of its characteristic accuracy and stability. Accuracy involves susceptibility to roundoff errors; stability involves accumulated roundoff errors not causing the algorithm to diverge. These are common problems when using the classical Kalman filter equations with large parameter and observation sets. The information form (synonymous with normal equations and opposite to the covariance form) was selected because: primary interest is in the smoothed results (in this implementation, filter state and covariance estimates never need to be computed); the smoother equations require the inverse of only an upper triangular matrix (of reduced size if covariance information is not required), instead of a full matrix (as in the RTS formulation); and the observations are assumed to already be edited. Editing can be done in a Kalman filter by comparing the predicted residual to the square root of its variance. However, these quantities are never computed explicitly in the information form. In fact, no predicted states and covariances are normally computed in this implementation.

The square root information filter/smoother (SRIF/S) is based on the equivalent concepts of a data equation and an information array as follows:

Data equation ≡ Information array

$$z = R\Delta x + v \qquad (R \;\; z) \tag{11}$$

where Δx = states to be estimated

R = nonsingular square matrix

v = zero-mean noise with unity (identity matrix) covariance

z = right-hand side of linear equations

Every data equation corresponds to an estimated state(Δx)-covariance (P) pair, i.e.,

$$\Delta x = R^{-1}z \quad \text{and} \quad P = R^{-1}R^{-T} \tag{12}$$

The information matrix (normal matrix) is $P^{-1} = R^T R$. Therefore, R^T is the square root of the information matrix--the origin of the name of this implementation. Data equations are not unique because if T is an orthogonal matrix

$$Tz = TR\Delta x + Tv \tag{13}$$

where Tv has unity covariance and

$$\Delta x = (TR)^{-1}Tz = R^{-1}T^{-1}Tz = R^{-1}z \tag{14}$$

Therefore, the transformed equations have the unity covariance noise term and the same solution. These results can be extended to the case where R has been augmented by additional rows representing new observations. In the SRIF/S method, Householder orthogonal transformations are used to partially or totally triangularize certain information arrays. Solutions, therefore, are always computed from an upper triangular set of linear equations. The Householder transformations are not computed explicitly, only the results of applying the transformations are needed. The details of applying these transformations are given in the **FILTER ALGORITHM** section. An excellent description of the square root information concepts and properties of the Householder transformations, on which the MSF/S development was based, is in Reference 4.

The parameter set selected for the MSF/S system can be divided into three categories:

1. Stochastic parameters (labelled p parameters)
   a. Orbit-related
   b. Measurement-related

2. Time-varying but nonstochastic parameters (labelled x parameters)

3. Bias parameters (labelled y parameters)
   a. Station-related
   b. Orbit-related

All parameters are arc parameters, i.e., there are no pass parameters. This grouping results in some simplification of the SRIF/S algorithms. Since all p parameters must be present twice in the propagation and array smoothing step arrays, not allowing all parameters to be stochastic results in smaller arrays. Also, the y parameters can be treated somewhat separately, resulting in additional array storage reduction. In addition, the Q matrix is only required for the p parameters.

Both the orbital element states and the clock states are treated as pseudoepoch state parameters. This means they are epoch state corrections that would have occurred had the process noise been zero. These can then be readily mapped to current state, using the standard state transition matrices. This definition was primarily adopted for two reasons: (1) to use the partial derivatives of position and velocity with respect to orbital elements generated in the integrator, as they would be used in a standard batch fit when forming the measurement matrices and (2) to reduce the clock model to a polynominal in time, with the fit epoch as the reference point if its process noise terms were zero. This simplifies the state equations and keeps the observation equations identical to those used for batch least squares.

Another concept employed in the MSF/S system development was to solve for a given parameter set using one set of observations, and then, use these solutions in processing another set of observations. The two specific parameter sets chosen were the orbit-related parameters and the satellite clock parameters. The orbit solution can be incorporated by using the propagated trajectories from a previous fit and not solving for any orbit-related parameters. The satellite clock solutions can be incorporated by using the total satellite clock offsets from a previous fit and not solving for satellite clock parameters. These two methods used together would allow stations, for which observations may or may not have been included in the previous fit, to be positioned with fixed orbits and satellite clocks as an evaluation procedure.

TIMELINE DEFINITIONS

Figure 3 gives a detailed diagram of the relationships of all the time-related quantities to be referred to in the rest of this report. All times are either GPS or UTC times.

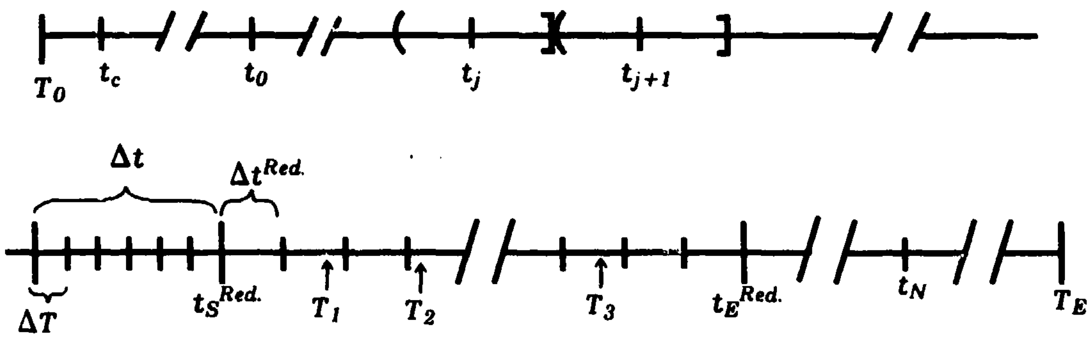


FIGURE 3. DETAILED TIMELINE DEFINITION

t_0 = start time of fit span
t_N = end time of fit span
(both) must be trajectory timelines

$\{t_j\}$ = mini-batch times

Δt = $t_{j+1} - t_j$ = mini-batch time step (integer multiple of trajectory time step)

( ] = mini-batch measurement limits

T_0 = trajectory epoch (may be different for each satellite), $T_0 \leq t_0$

T_E = end time of trajectory, $T_E \geq t_N$

ΔT = trajectory time step

t_c = clock epoch (may be different for each satellite and each station and may be before or after t_0)

$t_S^{Red.}$ = start time of reduced mini-batch step span (primarily for SATRACK), $t_S^{Red.} = t_0 + integer \times \Delta t$

$t_E^{Red.}$ = end time of reduced mini-batch step span, $t_E^{Red.} = t_S^{Red.} + integer \times \Delta t$

$\Delta t^{Red.}$ = reduced mini-batch time step, $\Delta t^{Red.} = \Delta t \,/\, integer$, $\Delta t^{Red.} = integer \times \Delta T$

$\{T_i\}_{i=1,2,3}$ = times for SATRACK covariance information

In addition each data type can be processed over a subspan of the fit span with the default being the entire fit span.

STATE EQUATIONS

The stochastic state equations in discrete form define the relationship between the states at t_j and t_{j+1}; i.e., the solutions that are determined by the Filter or Smoother must satisfy these equations. The general form of these equations was given in equation (1) in the **ESTIMATION CONCEPTS** section. A specialized form of these equations was selected for the MSF/S system to ensure efficient handling of the multisatellite capability and maximum utilization of available partial derivatives and other pre-computed quantities from the integrator. These equations are given by:

$$\begin{pmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{x} \\ \Delta \boldsymbol{y} \end{pmatrix}_{j+1} = \begin{pmatrix} M & 0 & 0 \\ V_{p_j} & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{x} \\ \Delta \boldsymbol{y} \end{pmatrix}_{j} + \begin{pmatrix} \boldsymbol{w}_j \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

where p, x, and y refer to categories of parameters.

$\Delta \boldsymbol{p}$ = stochastic parameter states–only states being driven by white noise

$\Delta \boldsymbol{x}$ = time-varying but nonstochastic parameter states

$\Delta \boldsymbol{y}$ = bias parameter states

$\boldsymbol{w}_j$ = white noise vector with covariance matrix $\boldsymbol{Q}_j$.

The matrix involving M, V_{p_j}, and the two identity matrices is the state transition matrix. The M matrix relates the p parameter states at t_j to t_{j+1}. The V_{p_j} matrix relates the p parameter states at t_j to the x parameter states at t_{j+1}. The p parameters are divided into two subcategories--orbit-related and measurement-related. The orbit-related p parameters are radiation pressure ($\boldsymbol{K_R}$) and gravitational accelerations ($\boldsymbol{G}$). The measurement-related p parameters are tropospheric refraction ($\boldsymbol{C_R}$), satellite clock ($\boldsymbol{C_{SV}}$), and station clock ($\boldsymbol{C_{MS}}$). The only x parameters are the pseudo-epoch orbital elements ($\boldsymbol{e}$). In addition, the y parameters are divided into two subcategories also--station-related and orbit-related. The only station-related y parameters are station coordinates ($\boldsymbol{S}$). The orbit-related y parameters are radiation pressure ($\boldsymbol{RP}$), thrust ($\boldsymbol{T}$), polar motion ($\boldsymbol{PM}$), and gravity coefficients ($\boldsymbol{GC}$). Polar motion is under the orbit-related category, instead of the station-related category, because the partial derivatives of satellite position with respect to polar motion can be nonzero. Parameters are present for either all satellites or all stations, depending on the specific parameter, except for thrusts that are only present for the appropriate satellites and polar motion and gravity coefficients, which are common to all satellites and are present once. Appendix A describes the assumptions and definitions made in adopting the specialized form of the state equations for the orbit-related parameters.

The state transition matrix M in equation (15) has the following block diagonal structure:

$$M = \begin{pmatrix} \begin{matrix} M_{K_{R_1}} & & 0 \\ & \ddots & \\ 0 & & M_{K_{R_{N_{SV}}}} \end{matrix} & 0 & 0 & 0 & 0 \\ 0 & \begin{matrix} M_{G_1} & & 0 \\ & \ddots & \\ 0 & & M_{G_{N_{SV}}} \end{matrix} & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} M_{C_{R_1}} & & 0 \\ & \ddots & \\ 0 & & M_{C_{R_{N_{MS}}}} \end{matrix} & 0 & 0 \\ 0 & 0 & 0 & \begin{matrix} M_{C_{SV_1}} & & 0 \\ & \ddots & \\ 0 & & M_{C_{SV_{N_{SV}}}} \end{matrix} & 0 \\ 0 & 0 & 0 & 0 & \begin{matrix} M_{C_{MS_1}} & & 0 \\ & \ddots & \\ 0 & & M_{C_{MS_{N_{MS}}}} \end{matrix} \end{pmatrix} \quad (16)$$

where N_{SV} = number of satellites and N_{MS} = number of stations.

The V_p matrix is also sparse and has the following structure, where each ϕ submatrix is of dimension 6×3:

$$V_p = \left(\begin{array}{cccc|cccc|c}
\Phi_{K_{R_1}} & 0 & & 0 & \Phi_{G_1} & 0 & & 0 & \\
0 & \Phi_{K_{R_2}} & & 0 & 0 & \Phi_{G_2} & & 0 & \\
 & & \ddots & & & & \ddots & & 0 \\
0 & & & \Phi_{K_{R_{NSV}}} & 0 & & & \Phi_{G_{NSV}} &
\end{array}\right) \tag{17}$$

The Q matrix corresponding to the p parameters is also sparse and has the following block diagonal structure:

$$Q = \begin{pmatrix} \begin{matrix} Q_{K_{R_1}} & 0 \\ & \ddots \\ 0 & Q_{K_{R_{NSV}}} \end{matrix} & 0 & 0 & 0 & 0 \\ 0 & \begin{matrix} Q_{G_1} & 0 \\ & \ddots \\ 0 & Q_{G_{NSV}} \end{matrix} & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} Q_{C_{R_1}} & 0 \\ & \ddots \\ 0 & Q_{C_{R_{NMS}}} \end{matrix} & 0 & 0 \\ 0 & 0 & 0 & \begin{matrix} Q_{C_{SV_1}} & 0 \\ & \ddots \\ 0 & Q_{C_{SV_{NSV}}} \end{matrix} & 0 \\ 0 & 0 & 0 & 0 & \begin{matrix} Q_{C_{MS_1}} & 0 \\ & \ddots \\ 0 & Q_{C_{MS_{NMS}}} \end{matrix} \end{pmatrix} \quad (18)$$

The Filter algorithm does not directly use the Q matrix; but rather, uses the square root of the inverse of the Q matrix R_w, i.e., $Q = R_w^{-1} R_w^{-T}$. R_w has the same block diagonal structure as Q except that each block is upper triangular.

The parameters are detailed below. It is assumed that the values solved for are corrections to the nominal values for all parameters. Included in the description of each parameter are the models assumed and a definition of the submatrices of M, V_p, and Q, if applicable. Also, units are given for each parameter. The x parameters are described first, so the submatrices of V_p can be defined for the p parameters as they are described. Several expressions involve Δt and must be re-evaluated for the reduced mini-batch span processing.

TIME-VARYING NONSTOCHASTIC STATES

The only x parameters are the pseudoepoch orbital elements at the trajectory epoch represented by $\bullet$. Each trajectory can have a different epoch; as long as it is before the fit epoch. These orbital elements are defined as follows:

a = semi-major axis (km)

$e \sin \omega$ = eccentricity × sine (argument of perigee) (unitless)

$e \cos \omega$ = eccentricity × cosine (argument of perigee) (unitless)

I = inclination (radians)

$M + \omega$ = mean anomaly + argument of perigee (radians)

Ω = right ascension of the ascending node (radians).

The inertial position r_0 and velocity $\dot{r}_0$ can alternatively be the x parameters, if partials with respect to these parameters are on the trajectory for any given satellite. The MSF/S system of programs is designed so that it does not matter which of these two parameter sets is chosen for a given satellite. Throughout the rest of this report o will be used to signify either set of orbit parameters. No direct process noise is included on the $\bullet$ parameters, nevertheless, they are smoothable as a result of being dynamically related (through the V_p matrix) to the K_R and G parameters, which are stochastic. Because of the pseudoepoch state formulation of the state equations (see Appendix A), the state transition matrix for these states is the identity matrix.

STOCHASTIC STATES

As mentioned above, the p parameters can be divided into two subcategories--orbit-related and measurement-related. Since all x parameters are orbit parameters, ordering the orbit-related p parameters first results in the V_p matrix being of the form given in equation (17).

Orbit-Related Stochastic States

Radiation Pressure

The 3 radiation pressure parameters, K_R , are

$$\begin{pmatrix} K_{R_1} \\ \\ K_{R_2} \\ \\ K_{R_3} \end{pmatrix} = \begin{pmatrix} \text{radiation pressure scale (unitless, .01} \approx \text{constant acceleration} \\ \text{of } 10^{-12} \text{ km/sec}^2 \text{ in satellite-sun direction)} \\ \text{y-axis acceleration } (10^{-12} \text{ km/sec}^2) \\ \text{angle between satellite x and y axes (radians, usually constrained} \\ \text{to 90 deg since it is not easily observed).} \end{pmatrix}$$

Nominal values for these parameters are those used in the integrator. The time history of the corrections to each of these parameters is modeled as a current state first-order Gauss-Markov process (see Appendix B). This results in the 3×3 portion of the state transition matrix M being given by

$$M_{K_R} = \begin{pmatrix} e^{-\frac{\Delta t}{\tau_{K_{R_1}}}} & 0 & 0 \\ 0 & e^{-\frac{\Delta t}{\tau_{K_{R_2}}}} & 0 \\ 0 & 0 & e^{-\frac{\Delta t}{\tau_{K_{R_3}}}} \end{pmatrix} \tag{19}$$

where $e^{-\frac{\Delta t}{\tau_{K_{R_i}}}}$ reduces to 1 for $\tau_{K_{R_i}} = \infty$, i.e., for a random walk process or bias. The process noise covariance matrix for these states is a diagonal matrix with each element given by

$$Q_{K_{R_{i,i}}} = q_{K_{R_i}} \frac{\tau_{K_{R_i}}}{2} \left(1 - e^{-\frac{2\Delta t}{\tau_{K_{R_i}}}} \right) \qquad i = 1, 2, 3 \tag{20}$$

where $q_{K_{R_i}}$ = spectral density of the white noise term = $\frac{2}{\tau_{K_{R_i}}} \sigma^2_{K_{R_i}}$ and $\sigma_{K_{R_i}}$ is the steady-state sigma of the process. These sigmas have the same units as the parameters. Each diagonal of the R_w matrix for these states is the square root of the inverse of each diagonal of the Q_{K_R} matrix. For a random walk process the spectral density must be directly input. This element of the process noise covariance matrix expression reduces to $q_{K_{R_i}} \Delta t$ for a random walk process and 0 for a bias, in which case the diagonal of R_w is set to a computational infinity. The 6×3 portion of the V_p matrix corresponding to K_R is given by

$$\Phi_{K_R} = \Phi_e^{-1}(t_{j+1}) \begin{pmatrix} \frac{\partial \mathbf{r}(t_{j+1})}{\partial \mathbf{K}_R(t_j)} \\ \frac{\partial \dot{\mathbf{r}}(t_{j+1})}{\partial \mathbf{K}_R(t_j)} \end{pmatrix} \tag{21}$$

where $\Phi_e(t_{j+1})$ = partials of position and velocity at time t_{j+1} with respect to epoch orbital elements obtained by interpolation from the trajectory.

Approximations using second order Taylor series expansions are used to obtain the partials of position and velocity at time t_{j+1} with respect to radiation pressure parameters at t_j. (This approximation is discussed in Appendix C.) These partials are given by

$$\frac{\partial \mathbf{r}(t_{j+1})}{\partial \mathbf{K}_R(t_j)} = \frac{\Delta t^2}{2} \frac{\partial \ddot{\mathbf{r}}(t_j)}{\partial \mathbf{K}_R(t_j)} \tag{22}$$

$$\frac{\partial \dot{\mathbf{r}}(t_{j+1})}{\partial \mathbf{K}_R(t_j)} = \Delta t \frac{\partial \ddot{\mathbf{r}}(t_j)}{\partial \mathbf{K}_R(t_j)} + \frac{\Delta t^2}{2} \frac{\partial \ddot{\mathbf{r}}(t_j)}{\partial \mathbf{K}_R(t_j)} B_{K_R} \tag{23}$$

where
$$B_{K_R} = \begin{pmatrix} -\frac{1}{\tau_{K_{R_1}}} & 0 & 0 \\ 0 & -\frac{1}{\tau_{K_{R_2}}} & 0 \\ 0 & 0 & -\frac{1}{\tau_{K_{R_3}}} \end{pmatrix} \qquad (24)$$

$= 0$ for a random walk or bias

$$\frac{\partial \ddot{r}(t_j)}{\partial K_R(t_j)} = R_s \begin{pmatrix} \frac{a_x - K_{R_2} 10^{12}\, shape \cos K_{R_3}}{K_{R_1}} & 10^{-12}\, shape \cos K_{R_3} & -K_{R_2} 10^{-12}\, shape \sin K_{R_3} \\ 0 & 10^{-12}\, shape \sin K_{R_3} & K_{R_2} 10^{-12}\, shape \cos K_{R_3} \\ \frac{a_z}{K_{R_1}} & 0 & 0 \end{pmatrix} \qquad (25)$$

a = inertial acceleration at t_j due to radiation pressure in the body-axes directions obtained from the trajectory

R_s = matrix required to transform between the body-axis and inertial Cartesian reference systems obtained from the trajectory at t_j

K_R = nominal radiation pressure parameter values from the trajectory

shape = fraction of the sun's disk unobstructed by any eclipsing body (Earth, Moon, or both) obtained from the trajectory at t_j

The effects of changes in ϵ on K_R are ignored because they are negligible. K_R parameters cannot be present without ϵ parameters. The partials of radiation pressure acceleration with respect to K_R will be 0 if t_j lies in the umbra region of the eclipse, even though some observations in the corresponding mini-batch interval may lie outside of the eclipse period. This form of mismodeling is not a problem when using these parameters in their intended stochastic manner.

Gravitational Accelerations

The 3 gravitational acceleration parameters, G, are the radial, along-track, and cross-track (RAC) accelerations in km/sec^2. They are called this because the primary error they are meant to absorb is gravity field model error. These parameters are sometimes referred to as unmodeled accelerations. The nominal value of each of these parameters is assumed to be zero. The time history of the corrections to each of these parameters is modeled as a current state first-order Gauss-Markov process (see Appendix B). This results in the 3×3 portion of the state transition matrix M being given by

$$M_G = \begin{pmatrix} e^{-\frac{\Delta t}{\tau_{G_1}}} & 0 & 0 \\ 0 & e^{-\frac{\Delta t}{\tau_{G_2}}} & 0 \\ 0 & 0 & e^{-\frac{\Delta t}{\tau_{G_3}}} \end{pmatrix} \tag{26}$$

where $e^{-\frac{\Delta t}{\tau_{G_i}}}$ reduces to 1 for $\tau_{G_i} = \infty$, i.e., for a random walk process or bias. The process noise covariance matrix for these states is a diagonal matrix with each element given by

$$Q_{G_{i,i}} = q_{G_i} \frac{\tau_{G_i}}{2}\left(1 - e^{-\frac{2\Delta t}{\tau_{G_i}}}\right) \qquad i = 1,2,3 \tag{27}$$

where q_{G_i} = spectral density of the white noise term $= \frac{2}{\tau_{G_i}} \sigma^2_{G_i}$ and σ_{G_i} is the steady-state sigma of the process. These sigmas have the same units as the parameters. Each diagonal of the R_w matrix for these states is the square root of the inverse of each diagonal of the Q_G matrix. For a random walk process the spectral density must be directly input. The process noise covariance matrix expression reduces to $q_{G_i}\Delta t$ for a random walk process and 0 for a bias, in which case the diagonal of R_w is set to a computational infinity. The 6×3 portion of the V_p matrix corresponding to G is given by

$$\Phi_G = \phi_e^{-1}(t_{j+1}) \begin{pmatrix} \dfrac{\partial r(t_{j+1})}{\partial G(t_j)} \\ \dfrac{\partial \dot{r}(t_{j+1})}{\partial G(t_j)} \end{pmatrix} \tag{28}$$

where $\phi_e(t_{j+1})$ = partials of position and velocity at time t_{j+1} with respect to epoch orbital elements obtained by interpolating off the trajectory.

Approximations using second order Taylor series expansions are used to obtain the partials of position and velocity at time t_{j+1} with respect to gravitational acceleration parameters at t_j. (This approximation is discussed in Appendix C.) These partials are given by

$$\frac{\partial r(t_{j+1})}{\partial G(t_j)} = \frac{\Delta t^2}{2} \frac{\partial \ddot{r}(t_j)}{\partial G(t_j)} \tag{29}$$

$$\frac{\partial \dot{r}(t_{j+1})}{\partial G(t_j)} = \Delta t \frac{\partial \ddot{r}(t_j)}{\partial G(t_j)} + \frac{\Delta t^2}{2} \frac{\partial \ddot{r}(t_j)}{\partial G(t_j)} B_G \tag{30}$$

where
$$B_G = \begin{pmatrix} -\frac{1}{\tau_{G_1}} & 0 & 0 \\ 0 & -\frac{1}{\tau_{G_2}} & 0 \\ 0 & 0 & -\frac{1}{\tau_{G_3}} \end{pmatrix} \tag{31}$$

$= 0$ for a random walk or bias

$$\frac{\partial \ddot{\vec{r}}(t_j)}{\partial G(t_j)} = \left(\hat{r} \quad \frac{\hat{r} \times \hat{v}}{|\hat{r} \times \hat{v}|} \times \hat{r} \quad \frac{\hat{r} \times \hat{v}}{|\hat{r} \times \hat{v}|} \right) \tag{32}$$

= matrix required to transform between the RAC and inertial Cartesian reference frames at t_j, where $\hat{}$ denotes a unit vector

The effects of changes in e on G are ignored because they are negligible. G parameters cannot be present without e parameters.

Measurement-Related Stochastic States

Tropospheric Refraction

The tropospheric refraction parameter, C_R, is the zenith tropospheric refraction parameter in km. The correction to this parameter is related to other elevations by a factor of $1/\sin E$. The time history of the correction to this parameter is modeled as a current state first-order Gauss-Markov process (see Appendix B). This results in the 1×1 portion of the state transition matrix M being given by

$$M_{C_R} = e^{-\frac{\Delta t}{\tau_{C_R}}} \tag{33}$$

where $e^{-\frac{\Delta t}{\tau_{C_R}}}$ reduces to 1 for $\tau_{C_R} = \infty$, i.e., for a random walk process or bias. The process noise covariance matrix for this state consists of a single element given by

$$Q_{C_R} = q_{C_R} \frac{\tau_{C_R}}{2} \left(1 - e^{-\frac{2\Delta t}{\tau_{C_R}}} \right) \tag{34}$$

where q_{C_R} = spectral density of the white noise term $= \frac{2}{\tau_{C_R}} \sigma^2_{C_R}$ and σ_{C_R} is the steady-state sigma of the process. This sigma has the same unit as the parameter. The R_w matrix for this state is the square root of the inverse of Q_{C_R}. For a random walk process the spectral density must be directly input. The process noise covariance matrix expression reduces to $q_{C_R} \Delta t$ for a random walk process and 0 for a bias, in which case R_w is set to a computational infinity.

Satellite and Station Clocks

The clock models for the satellite and station clocks are given below. Appendix D discusses the models for the satellite and station clocks in more detail than given here and describes the relationship between the clock model spectral density noise terms and the Allan variance. The frequency offset term in these models is not the instantaneous frequency offset, since it does not contain the white frequency noise component. This component is only observable by its integrated effect on the time offset. These clock models reduce to polynomials referenced to the fit epoch when all the process noise terms are zero.

The satellite clock parameters, C_{SV}, are

$$C_{SV} = \begin{pmatrix} \Delta\ddot{\iota}_0 \\ \Delta\dot{\iota}_0 \\ \Delta\iota_0 \end{pmatrix} = \begin{pmatrix} \text{frequency drift (ppm/sec)} \\ \text{frequency offset (ppm)} \\ \text{time offset } (\mu\text{sec}) \end{pmatrix}$$

The nominal values for these parameters are described in the next section and are based on initial polynomials and step changes. The clock model is implemented in a pseudoepoch state form which results in the portion of the state transition matrix corresponding to these states being an identity matrix, i.e.,

$$M_{C_{SV}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{35}$$

The state noise covariance matrix for this set of states is given by

$$Q_{U_{C_{SV}}} = \Phi_{C_{SV}}^{-1} \begin{pmatrix} q_1\Delta t & q_1\frac{\Delta t^2}{2} & q_1\frac{\Delta t^3}{6} \\ q_1\frac{\Delta t^2}{2} & q_1\frac{\Delta t^3}{3} + q_2\Delta t & q_1\frac{\Delta t^4}{8} + q_2\frac{\Delta t^2}{2} \\ q_1\frac{\Delta t^3}{6} & q_1\frac{\Delta t^4}{8} + q_2\frac{\Delta t^2}{2} & q_1\frac{\Delta t^5}{20} + q_2\frac{\Delta t^3}{3} + q_3\Delta t \end{pmatrix} \Phi_{C_{SV}}^{-T} = \Phi_{C_{SV}}^{-1} Q_{C_{SV}} \Phi_{C_{SV}}^{-T} \tag{36}$$

where

$$\Phi_{C_{SV}} = \begin{pmatrix} 1 & 0 & 0 \\ t_{j+1} - t_0 & 1 & 0 \\ \frac{(t_{j+1} - t_0)^2}{2} & t_{j+1} - t_0 & 1 \end{pmatrix} \tag{37}$$

$\{q_i\}_{i=1,2,3}$ = white noise spectral density values in units of (ppm/sec)^2 /sec, (ppm)^2/sec, and (μsec)^2 /sec; and $t_{j+1} - t_0$ is in seconds.

The Filter actually requires $Q^{-1}_{C_{SV}} = Q^{\prime -1}_{C_{SV}} \Phi_{C_{SV}}$. To get this matrix $Q^{\prime}_{C_{SV}}$ is factored into $R^T_{C_{SV}} R_{C_{SV}}$ using lower triangular Cholesky decomposition (see Appendix E). Then, $R_{C_{SV}} \Phi_{C_{SV}}$ is no longer upper triangular and must be upper triangularized before being used in the propagation step for computational simplifications. If some of the q_i values are zero, special processing must be done to ensure that $Q_{C_{SV}}$ is nonsingular. If $q_1 \neq 0$ and q_2, q_3 or both $= 0$, no change is necessary. If $q_1 = 0$, $Q_{1,1}$ is set to ≈ 0. If $q_1 = q_2 = 0$, $Q_{2,2}$ is set to ≈ 0. If $q_1 = q_2 = q_3 = 0$, $Q_{3,3}$ is set to ≈ 0. If only processing range difference data, each q_3 is set to ≈ 0. However, the *a priori* sigmas on the parameters should not be set to ≈ 0. Because the time offset state is basically the integral of the frequency offset state, constraining the time offset also constrains the frequency offset, which is not desired.

For certain clock events the $Q_{C_{SV}}$ matrix is changed for the mini-batch step propagation that contains the event and then changed back to its original values. This is used to account for the uncertainty in the clock event input offsets. For a C-field adjust, $R^{\prime}_{C_{SV}}$ replaces $R_{C_{SV}}$, where

$$R^{\prime}_{C_{SV}} = Q^{\prime -1}_{C_{SV}} \tag{38}$$

and

$$Q^{\prime}_{C_{SV}} = Q_{C_{SV}} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^2_{\Delta \dot{\iota}_{C\text{-}field}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{39}$$

$\sigma_{\Delta \dot{\iota}_{C\text{-}field}}$ is input in parts in 10^{12} and converted to ppm before use. Z-count adjusts are assumed to be exact, so there is no process noise adjustment associated with this event. For a clock reinitialization event, $R^{\prime\prime}_{C_{SV}}$ replaces $R_{C_{SV}}$, where

$$R^{\prime\prime}_{C_{SV}} = \begin{pmatrix} \frac{1}{\sigma_{\Delta \ddot{\iota}_0}} & 0 & 0 \\ 0 & \frac{1}{\sigma_{\Delta \dot{\iota}_0}} & 0 \\ 0 & 0 & \frac{1}{\sigma_{\Delta \iota_0}} \end{pmatrix} \tag{40}$$

and each sigma is the *a priori* sigma used in the Filter initialization step. For a frequency change event, $R^{\prime\prime\prime}_{C_{SV}}$ replaces $R_{C_{SV}}$, where

$$R^{\prime\prime\prime}_{C_{SV}} = Q^{\prime\prime\prime -1}_{C_{SV}} \tag{41}$$

and
$$Q'''_{C_{SV}} = Q_{C_{SV}} + \begin{pmatrix} \sigma^2_{\Delta\ddot{\tau}_{fc}} & 0 & 0 \\ 0 & \sigma^2_{\Delta\dot{\tau}_{fc}} & 0 \\ 0 & 0 & \sigma^2_{\Delta\tau_{fc}} \end{pmatrix} \tag{42}$$

where each input sigma is converted to the proper units before use.

The station clock parameters, C_{MS}, are

$$C_{MS} = \begin{pmatrix} \Delta\dot{\iota}_0 \\ \Delta\iota_0 \end{pmatrix} = \begin{pmatrix} \text{frequency offset (ppm)} \\ \text{time offset } (\mu\text{sec}) \end{pmatrix}$$

This model is identical to the satellite clock model, except the frequency drift state is absent. The nominal values are also based on an initial polynomial and step changes. The model is implemented in a pseudoepoch state form, which results in the portion of state transition matrix corresponding to these states being an identity matrix, i.e.,

$$M_{C_{MS}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{43}$$

The state noise covariance matrix for this set of states is given by

$$Q_{0C_{MS}} = \Phi^{-1}_{C_{MS}} \begin{pmatrix} q_1\Delta t & q_1 \dfrac{\Delta t^2}{2} \\ q_1 \dfrac{\Delta t^2}{2} & q_1 \dfrac{\Delta t^3}{3} + q_2\Delta t \end{pmatrix} \Phi^{-T}_{C_{MS}} = \Phi^{-1}_{C_{MS}}\, Q_{C_{MS}}\, \Phi^{-T}_{C_{MS}} \tag{44}$$

where
$$\Phi_{C_{MS}} = \begin{pmatrix} 1 & 0 \\ t_{j+1} - t_0 & 1 \end{pmatrix} \tag{45}$$

$\{q_i\}_{i=1,2}$ = white noise spectral densities in units of (ppm)²/sec and (μsec)²/sec; and $t_{j+1} - t_0$ is in seconds

The Filter actually requires $Q^{-1}_{0C_{MS}} = Q^{-1}_{C_{MS}} \Phi_{C_{MS}}$. To get this matrix $Q^{-1}_{C_{MS}}$ is first factored into $R^T_{C_{MS}}\, R_{C_{MS}}$ using lower triangular Cholesky decomposition (see Appendix E). Then, $R_{C_{MS}} \Phi_{C_{MS}}$ is no longer upper triangular and must be upper triangularized before being used in the propagation step for computational simplifications. If some of the q_i values are zero, special processing must be done to ensure that $Q_{C_{MS}}$ is nonsingular. If $q_1 = 0$, $Q_{1,1}$ is set to ≈ 0. If in addition $q_2 = 0$, $Q_{2,2}$ is set to ≈ 0. If only processing range difference data, each q_2 is set to ≈ 0. However, the *a priori* sigmas on the parameters should not be set to ≈ 0. This is because the time offset state is basically the integral of the frequency offset state. Constraining the time offset also constrains the frequency offset, which is not desired. For the selected master station the q_i values are set to ≈ 0.

For certain clock events, the $Q_{C_{MS}}$ matrix is changed for the mini-batch step propagation that contains the event and then changed back to its original values. This is used to account for the uncertainty in the clock event input offsets. For a clock reinitialization, $R'_{C_{MS}}$ replaces $R_{C_{MS}}$, where

$$R'_{C_{MS}} = \begin{pmatrix} \frac{1}{\sigma_{\Delta \dot{\iota}_0}} & 0 \\ 0 & \frac{1}{\sigma_{\Delta \iota_0}} \end{pmatrix} \tag{46}$$

and each sigma is the *a priori* sigma used in the Filter initialization. This is not done for a master station clock reinitialization, which is intended to accomodate GPS time steering. For a frequency change event, $R''_{C_{MS}}$ replaces $R_{C_{MS}}$, where

$$R''_{C_{MS}} = Q''^{-1}_{C_{MS}} \tag{47}$$

and

$$Q''_{C_{MS}} = Q_{C_{MS}} + \begin{pmatrix} \sigma^2_{\Delta \dot{\iota}_{fc}} & 0 \\ 0 & \sigma^2_{\Delta \iota_{fc}} \end{pmatrix} \tag{48}$$

where each input sigma is converted to the proper units before use. For a master station switch event, the $R_{C_{MS}}$ matrix for the station that is no longer the master station is replaced by

$$R'''_{C_{MS}} = \begin{pmatrix} \frac{1}{\sigma_{\Delta \dot{\iota}_0}} & 0 \\ 0 & \frac{1}{\sigma_{\Delta \iota_0}} \end{pmatrix} \tag{49}$$

for the first propagation step for which $t_{j+1} > t_{MSS}$. Then, the matrix is reset to the $R_{C_{MS}}$ values originally computed from input and saved (before it was replaced by a matrix based on q_i's set to ≈ 0) for propagation from t_{j+1} to t_{j+2} and all subsequent steps. The master station $Q_{C_{MS}}$ matrix (essentially a null matrix) is then used from t_j to t_{j+1} and all subsequent propagation steps for the new master station parameters. No change in the process noise is required for a station time change event, since it is assumed to be exact.

BIAS STATES

As previously mentioned, the y parameters are divided into two subcategories–station-related and orbit-related. All orbit-related y parameters require partial derivatives from the trajectories except polar motion, in which case partials are present only if the geopotential expansion axis is not the instantaneous spin axis or the Celestial Ephemeris Pole.

<u>Station-Related Bias States</u>

<u>Station Coordinates</u>

The corrections to station coordinates, ΔS, are defined in a local-vertical reference frame as follows:

$$\Delta S = \begin{pmatrix} \Delta E \\ \Delta N \\ \Delta V \end{pmatrix} = \begin{pmatrix} \text{east component} \\ \text{north component} \\ \text{vertical component} \end{pmatrix} (\text{km})$$

The nominal station coordinates are input in terms of longitude, latitude, and height referenced to a specified ellipsoid. The state transition matrix for these states is the identity matrix.

<u>Orbit-Related Bias States</u>

This category of parameter is included so constant force model parameters affecting the orbit can be solved for. All of these parameter sets have identity state transition matrices; and because of the pseudoepoch orbital element state definition, they result in no terms that relate changes in e to changes in these parameters. Partial derivatives are only required in forming the measurement condition equations and are obtained by interpolating off the trajectory. Four parameter sets fall in this category.

<u>Radiation Pressure</u>

The 3 radiation pressure parameters, RP, are

$$\begin{pmatrix} RP_1 \\ RP_2 \\ RP_3 \end{pmatrix} = \begin{pmatrix} \text{radiation pressure scale (unitless)} \\ \text{y-axis acceleration } (10^{-12}\ \text{km/sec}^2) \\ \text{angle between the x and y axes (radians are usually} \\ \text{constrained to 90 deg, since it is not easily observed)} \end{pmatrix}$$

These parameters are present for every satellite, if selected, and are identical to the stochastic radiation pressure parameters except the corrections are modeled as epoch state constants. The stochastic parameter states can be configured to emulate current state constants. However, this method is not recommended because of the partial derivative approximations made and the fact that the smoothing procedure would have to be completed to obtain the orbit solution at each timeline. The RP parameters should be used in two cases: to emulate a batch orbit fit using the Filter or to estimate constants along with their stochastic counterparts (since the latter states are assumed to be of zero mean).

Thrust

The thrust parameters, $\boldsymbol{T}$, are

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} \text{first component} \\ \text{second component} \\ \text{third component} \end{pmatrix} \quad (\text{km/sec}^2)$$

Thrusts can be modeled in the integrator in one of three reference frames–the body-axis frame, the RAC frame, or the RVC frame (see Reference 1). Thrust parameters are only present for specific satellites, as required, and their nominal values are obtained from the trajectories.

Polar Motion

The polar motion parameters, $\boldsymbol{PM}$, are

$$\begin{pmatrix} p \\ q \\ \dot{\Delta t} \end{pmatrix} = \begin{pmatrix} \text{pole component along Greenwich meridian (radians)} \\ \text{pole component along meridian 90 deg west of Greenwich (radians)} \\ \text{rate of change of UT1-UTC (sec/sec)} \end{pmatrix}$$

The corrections to these parameters are modeled as constants over the entire fit span with the $\dot{\Delta t}$ term reference time being the fit epoch. These states are common to all satellites and stations.

Gravity Coefficients

The gravity coefficient parameters, $\boldsymbol{GC}$, are selected gravity field model coefficients (unitless). These parameters are common to all satellites and the nominal values are those used by the integrator.

When the orbit-related parameters are viewed together, several comments apply. With only one set of orbit-related stochastic parameter states and pseudo-epoch orbital element states being solved for, the orbit model is essentially equivalent to solving for position, velocity, and acceleration corrections with the acceleration corrections constrained to be a zero-mean Gauss-Markov or random walk process. The $\boldsymbol{K_R}$ and $\boldsymbol{G}$ parameters should not be used as stochastic parameters simultaneously. If $\boldsymbol{K_R}$ is chosen, it can be viewed as solving for acceleration corrections along the satellite-sun line and along the y axis. This is because the direct radiation pressure force is almost constant throughout the orbit, so the scale parameter is simply scaling this near-constant acceleration. If $\boldsymbol{G}$ parameters are chosen, the acceleration solved for is resolved into the RAC coordinate frame. The constant $\boldsymbol{RP}$ parameters could be used with the stochastic $\boldsymbol{K_R}$ parameters to account for non-zero mean accelerations in the satellite-sun and y-axis directions. A continuous thrust would serve the same purpose for the $\boldsymbol{G}$ parameters.

OBSERVATION EQUATIONS AND PARTIAL DERIVATIVES

RANGE

As mentioned in the **GPS DATA FLOW** section it is assumed that all data to be used in the MSF/S system have been corrected for time transmission, relativity, satellite antenna offset, tropospheric refraction, and solid earth tide station height effects. Therefore the observation model is considerably simplified. For range (R) data the nonlinear observation equation is given by

$$R_{i,k}(t_{obs}) = \left| r_i(t_{obs}) - r_k(t_{obs}) \right| - \frac{c}{10^6}\left[\Delta t_i^0(t_{obs}) + \Delta t_i(t_{obs}) \right] + \frac{c}{10^6}\left[\Delta t_k^0(t_{obs}) + \Delta t_k(t_{obs}) \right] + \frac{\Delta C_{R_k}(t_{obs})}{\sin E_{i,k}(t_{obs})} \tag{50}$$

where i = satellite subscript

k = station subscript

t_{obs} = observation time

$r_i(t_{obs})$ = inertial coordinates of satellite i at t_{obs} interpolated off the appropriate trajectory

$$r_k(t_{obs}) = [ABCD(t_{obs})]^T r_{k_{EF}} \tag{51}$$

= inertial coordinates of station k at t_{obs}

$$r_{k_{EF}} = (A' + h_k)\begin{pmatrix} \cos\phi_k \cos\lambda_k \\ \cos\phi_k \sin\lambda_k \\ \sin\phi_k \end{pmatrix} - A'e^2\begin{pmatrix} 0 \\ 0 \\ \sin\phi_k \end{pmatrix} \tag{52}$$

= Cartesian Earth-fixed station coordinates

(λ_k, ϕ_k, h_k) = geodetic coordinates of station k

$\frac{1}{f}$ = oblateness of Earth

$$e = [(2-f)f]^{\frac{1}{2}} \tag{53}$$

$$A' = \frac{a_{Earth}}{(1 - e^2 \sin^2\phi_k)^{\frac{1}{2}}} \tag{54}$$

a_{Earth} = semimajor axis of Earth's reference ellipsoid

$ABCD(t_{obs})$ = inertial-to-Earth-fixed rotation matrix at t_{obs}. (Reference 1 contains the details of the inertial-to-Earth-fixed transformation computations.)

c = speed of light (km/sec)

$\Delta\iota_i^0(t_{obs})$ = nominal clock time offset of the ith satellite clock from GPS time at t_{obs} (to be described below)

$\Delta\iota_i(t_{obs})$ = time offset correction to the nominal clock for the ith satellite at t_{obs} (nominally zero)

$\Delta\iota_k^0(t_{obs})$ = nominal clock time offset of the kth station clock from GPS time at t_{obs} (to be described below)

$\Delta\iota_k(t_{obs})$ = time offset correction to the nominal clock for the kth station at t_{obs} (nominally zero)

$\Delta C_{R_k}(t_{obs})$ = zenith tropospheric refraction correction at t_{obs} (nominally zero)

$$E_{i,k}(t_{obs}) = 90^\circ - Arccos(\hat{\rho}\cdot\hat{u}_V) \tag{55}$$

= instantaneous elevation angle from station k to satellite i

$$\rho = r_i(t_{obs}) - r_k(t_{obs}) \tag{56}$$

$$u_V = [ABCD(t_{obs})]^T \begin{pmatrix} x'_{EF} \\ y'_{EF} \\ z'_{EF}/1-e^2 \end{pmatrix} \tag{57}$$

The vector $\begin{pmatrix} x'_{EF} \\ y'_{EF} \\ z'_{EF} \end{pmatrix}_k$ is computed by evaluating equation (52) with $h_k = 0$, and $\hat{}$ indicates a unit vector.

Computation of $\Delta\iota_i^0(t)$ and $\Delta\dot{\iota}_i^0(t)$

The nominal satellite time and frequency offsets at an arbitrary time t are given by

$$\Delta\iota_i^0(t) = \Delta\iota_{0_i} + \Delta\dot{\iota}_{0_i}(t - t_{c_i}) + \Delta\ddot{\iota}_{0_i}\frac{(t-t_{c_i})^2}{2} \tag{58}$$

$$\Delta\dot{\iota}_i^0(t) = \Delta\dot{\iota}_{0_i} + \Delta\ddot{\iota}_{0_i}(t - t_{c_i}) \tag{59}$$

where $\Delta\iota_{0_i}$, $\Delta\dot{\iota}_{0_i}$, and $\Delta\ddot{\iota}_{0_i}$ are input quantities converted to internal units (nsec→µsec, parts in 10^{12}→ppm, and parts in 10^{12}/day→ppm/sec) and t_{c_i} is the ith satellite's clock epoch. To accomodate jumps in the nominal clocks, four satellite clock events have been defined. These events are described below in terms of their effects on the nominal clocks. The corresponding clock process noise adjustments were discussed in the previous section. All events are input as time of the event (in day number and seconds of the day) and associated clock offsets (in nsec, parts in 10^{12}, and parts in 10^{12}/day). These offsets are converted to the above internal units before use. Processing of each event is described below.

1. C-field adjust

A C-field adjust is a generic name for a commanded change in the satellite's clock frequency (usually only applies to Cesium clocks). The amount of the change is not known exactly, so a frequency uncertainty is added to the process noise matrix for this event (see the **STATE EQUATIONS** section). This event affects the nominal clock as follows:

For the first time $t > t_{C\text{-}field_i}$

$$\Delta\iota_{0_i} \text{ is replaced by } \Delta\iota_{0_i} + \Delta\dot{\iota}_{0_i}(t_{C\text{-}field_i} - t_{c_i}) + \Delta\ddot{\iota}_{0_i}\frac{(t_{C\text{-}field_i} - t_{c_i})^2}{2}$$

and $\Delta\dot{\iota}_{0_i}$ is replaced by $\Delta\dot{\iota}_{0_i} + \Delta\dot{\iota}_{C\text{-}field_i} + \Delta\ddot{\iota}_{0_i}(t_{C\text{-}field_i} - t_{c_i})$

All subsequent nominal clock computations for this satellite are referenced to a redefined epoch, $t_{C\text{-}field_i}$, and use equations (58) and (59).

2. Z-count adjust

A Z-count adjust is a commanded change in the satellite clock's time offset. The amount of the change is known exactly. This event affects the nominal clock as follows:

For the first time $t \geq t_{Z\text{-}count_i}$

$\Delta\iota_{0_i}$ is replaced by $\Delta\iota_{0_i} + \Delta\iota_{Z\text{-}count_i}$

All further nominal clock computations for this satellite are based on equations (58) and (59) with the clock epoch unchanged.

3. Clock reinitialization

A clock reinitialization is either a switch in the operational clock on the satellite or an anomalous phase jump in the current clock. The uncertainties in the clock estimates are set back to approximately the initialization values (see **STATE EQUATIONS** section). This event affects the nominal clock as follows:

For the first time $t \geq t_{reinit_i}$

$\Delta\iota_{0_i}$, $\Delta\dot{\iota}_{0_i}$, and $\Delta\ddot{\iota}_{0_i}$ are replaced by a new set of values from input and the clock epoch is redefined to t_{reinit_i} for all further nominal clock computations.

4. Frequency change

A frequency change is an unexplained jump in frequency. For this event, it is possible to increase the uncertainty in each clock state (see **STATE EQUATIONS** section). This event affects the nominal clock as follows:

For the first time $t > t_{fc_i}$

$$\Delta\iota_{0_i} \text{ is replaced by } \Delta\iota_{0_i} + \Delta\dot{\iota}_{0_i}(t_{fc_i} - t_{c_i}) + \Delta\ddot{\iota}_{0_i}\frac{(t_{fc_i} - t_{c_i})^2}{2}$$

$\Delta\dot{\iota}_{0_i}$ is replaced by $\Delta\dot{\iota}_{fc_i}$

and $\Delta\ddot{\iota}_{0_i}$ is replaced by $\Delta\ddot{\iota}_{fc_i}$

All subsequent nominal clock computations for this satellite are referenced to a redefined epoch, t_{fc_i}, and use equations (58) and (59).

If the total satellite clock offsets from a previous Filter or Smoother run are available,

$$\Delta\iota_i^{0}(t_{obs}) + \Delta\iota_i(t_{obs}) \text{ is replaced by } \Delta\iota_{Total_i}(t_j) + \Delta\dot{\iota}_{Total_i}(t_j)\,(t_{obs} - t_j)$$

where $t_j - \dfrac{\Delta t}{2} < t_{obs} \le t_j + \dfrac{\Delta t}{2}$

The total satellite clock offsets are defined in the **SOLUTION AND DIAGNOSTICS** section.

Computation of $\Delta\iota_k^{0}(t)$ and $\Delta\dot{\iota}_k^{0}(t)$

The nominal station time and frequency offsets at an arbitrary time t are given by

$$\Delta\iota_k^{0}(t) = \Delta\iota_{0_k} + \Delta\dot{\iota}_{0_k}(t - t_{c_k}) + \Delta\ddot{\iota}_{0_k}\,\frac{(t - t_{c_k})^2}{2} \tag{60}$$

$$\Delta\dot{\iota}_k^{0}(t) = \Delta\dot{\iota}_{0_k} + \Delta\ddot{\iota}_{0_k}(t - t_{c_k}) \tag{61}$$

where $\Delta\iota_{0_k}$, $\Delta\dot{\iota}_{0_k}$, and $\Delta\ddot{\iota}_{0_k}$ are input quantities converted to internal units (nsec→μsec, parts in 10^{12}→ppm, and parts in 10^{12}/day→ppm/sec) and t_{c_k} is the kth station's clock epoch. To accomodate jumps in the nominal clocks, three station clock events have been defined. These events are described below in terms of their effects on the nominal clocks. The corresponding clock process noise adjustments were discussed in the previous section. All events are input as time of the event (in day number and seconds of the day) and associated clock offsets (in nsec, parts in 10^{12}, and parts in 10^{12}/day). These offsets are converted to the internal units before use. Processing of each event is described below.

1. Clock reinitialization

A clock reinitialization is either a switch in the operational clock at the station or an anomalous phase jump in the current clock. The uncertainties in the clock estimates are set back to approximately their initialization values (see **STATE EQUATIONS** section). This event affects the nominal clock as follows:

For the first time $t \ge t_{reinit_k}$

$\Delta\iota_{0_k}$, $\Delta\dot{\iota}_{0_k}$, and $\Delta\ddot{\iota}_{0_k}$ are replaced by a new set of values from input and the clock epoch is redefined to t_{reinit_k} for all further nominal clock computations. This event is also used to accommodate the GPS time steering procedure when the offsets between GPS time and the master clock time change, but the designated master clock does not.

2. Frequency change

A frequency change is an unexplained jump in frequency. For this event, it is possible to increase the uncertainty in each clock state (see **STATE EQUATIONS** section). This event affects the nominal clock as follows:

For the first time $t > t_{fc_k}$

$\Delta\tau_{0_k}$ is replaced by $\Delta\tau_{0_k} + \Delta\dot{\tau}_{0_k}(t_{fc_k} - t_{c_k}) + \Delta\ddot{\tau}_{0_k}\dfrac{(t_{fc_k} - t_{c_k})^2}{2}$

$\Delta\dot{\tau}_{0_k}$ is replaced by $\Delta\dot{\tau}_{fc_k}$

and $\Delta\ddot{\tau}_{0_k}$ is replaced by $\Delta\ddot{\tau}_{fc_k}$

All subsequent nominal clock computations for this station are referenced to a redefined epoch, t_{fc_k}, and use equations (60) and (61).

3. Time change

A time change is a commanded change in the station clock's time offset. The amount of the change is known exactly. This event affects the nominal clock as follows:

For the first time $t \geq t_{tc}$

$\Delta\tau_{0_k}$ is replaced by $\Delta\tau_{0_k} + \Delta\tau_{tc_k}$

All subsequent nominal clock computations for this satellite are based on equations (60) and (61) with the clock epoch unchanged.

4. Master station switch

The master station switch event involves two stations simultaneously: the original and the new master stations. For the station that is no longer the master, a clock reinitialization is done at t_{MSS}, i.e.,

$\Delta\tau_{0_k}$, $\Delta\dot{\tau}_{0_k}$, and $\Delta\ddot{\tau}_{0_k}$ are replaced by new input quantities,

the clock epoch is redefined to be t_{MSS}, and all subsequent nominal clock computations are referenced to this redefined epoch. For the new master station

$\Delta\tau_{0_{k'}}$, $\Delta\dot{\tau}_{0_{k'}}$, and $\Delta\ddot{\tau}_{0_{k'}}$ are replaced by new input quantities,

the clock epoch is redefined to be t_{MSS}, and all subsequent nominal clock computations for this station are referenced to this redefined epoch. The process noise terms for these stations' clocks are adjusted as specified in the previous section. In addition, the information array elements corresponding to the clock states for the new master station have to be modified. (See the **FILTER ALGORITHM** section.)

The linear measurement model, equation (2), requires computation of the measurement residuals, z, and partial derivatives of the observations with respect to the parameters, A. Each residual is computed by evaluating the observation equation (50) at t_{obs}, with $\Delta\tau_i(t_{obs})$, $\Delta\tau_k(t_{obs})$, and $\Delta C_{R_i}(t_{obs})$ set to zero and subtracting the result from the observed value. Next, the partial derivatives required for the A matrix are defined. Whitening and decorrelation of the observations are discussed in the **FILTER ALGORITHM** section.

Range Partial Derivatives

As mentioned in the **ESTIMATION CONCEPTS** section, the partial derivatives of the observation with respect to the parameters are required for the Filter algorithm. These derivatives are obtained by differentiating the observation equation with respect to all parameters explicitly present, and using the chain rule and models to relate them to the solution parameter states. Let $\boldsymbol{\rho}_{i,k} = \mathbf{r}_i - \mathbf{r}_k$, $\rho_{i,k} = |\boldsymbol{\rho}_{i,k}|$, and $R = R_{i,k}$, then the partial derivatives of range with respect to the parameters in the observation equation are:

$$\frac{\partial R}{\partial \mathbf{r}_i} = -\frac{\partial R}{\partial \mathbf{r}_k} = \frac{\boldsymbol{\rho}^r_{i,k}}{\rho_{i,k}} \tag{62}$$

$$\frac{\partial R}{\partial \Delta\iota_i} = -\frac{\partial R}{\partial \Delta\iota_k} = -\frac{c}{10^6} \tag{63}$$

$$\frac{\partial R}{\partial \Delta C_{R_k}} = \frac{1}{\sin E_{i,k}} \tag{64}$$

Using these partial derivatives, the partial derivatives with respect to the solution parameters are:

$$\frac{\partial R}{\partial K_{R_i}} = \frac{\partial R}{\partial \mathbf{r}_i}\frac{\partial \mathbf{r}_i}{\partial K_{R_i}} = \frac{\partial R}{\partial \mathbf{r}_i}\frac{(t_{obs}-t_j)^2}{2}\frac{\partial \ddot{\mathbf{r}}_i(t_j)}{\partial K_{R_i}(t_j)} \tag{65}$$

(see equations (22) and (25))

$$\frac{\partial R}{\partial \mathbf{G}_i} = \frac{\partial R}{\partial \mathbf{r}_i}\frac{\partial \mathbf{r}_i}{\partial \mathbf{G}_i} = \frac{\partial R}{\partial \mathbf{r}_i}\frac{(t_{obs}-t_j)^2}{2}\frac{\partial \ddot{\mathbf{r}}_i(t_j)}{\partial \mathbf{G}_i(t_j)} \tag{66}$$

(see equations (29) and (32))

$$\frac{\partial R}{\partial \Delta C_{R_k}(t_j)} = \frac{\partial R}{\partial \Delta C_{R_k}}\frac{\partial \Delta C_{R_k}}{\partial \Delta C_{R_k}(t_j)} = \frac{\partial R}{\partial \Delta C_{R_k}}\, e^{-\frac{(t_{obs}-t_j)}{\tau_{C_{R_k}}}} \tag{67}$$

$$\frac{\partial R}{\partial \mathbf{C}_{SV_i}} = \frac{\partial R}{\partial \Delta\iota_i}\frac{\partial \Delta\iota_i}{\partial \mathbf{C}_{SV_i}} = \frac{\partial R}{\partial \Delta\iota_i}\left(\frac{(t_{obs}-t_0)^2}{2} \quad t_{obs}-t_0 \quad 1\right) \tag{68}$$

$$\frac{\partial R}{\partial \mathbf{C}_{MS_k}} = \frac{\partial R}{\partial \Delta\iota_k}\frac{\partial \Delta\iota_k}{\partial \mathbf{C}_{MS_k}} = \frac{\partial R}{\partial \Delta\iota_k}\left(t_{obs}-t_0 \quad 1\right) \tag{69}$$

$$\frac{\partial R}{\partial \mathbf{e}_i} = \frac{\partial R}{\partial \mathbf{r}_i}\frac{\partial \mathbf{r}_i}{\partial \mathbf{e}_i} \tag{70}$$

where $\frac{\partial r_i}{\partial e_i}$ is interpolated off the reference trajectory at t_{obs}

$$\frac{\partial R}{\partial \Delta S_k} = \frac{\partial R}{\partial r_k} \frac{\partial r_k}{\partial \Delta S_k} = \frac{\partial R}{\partial r_k} (ABCD)^T T_k \tag{71}$$

where $$T_k = (\hat{u}_E \; \hat{u}_N \; \hat{u}_V)_k \tag{72}$$

$$u_0 = \begin{pmatrix} x'_{kF} \\ y'_{kF} \\ 0 \end{pmatrix} \tag{73}$$

$$u_V = \begin{pmatrix} x'_{kF} \\ y'_{kF} \\ z'_{kF}/1-e^2 \end{pmatrix}, \hat{u}_V = \frac{u_V}{|u_V|} \tag{74}$$

$$\hat{u}_E = \frac{u_V \times u_0}{|u_V \times u_0|} \quad (-if \, \phi_k < 0^\circ) \tag{75}$$

$$u_N = u_V \times \hat{u}_E, \; \hat{u}_N = \frac{u_N}{|u_N|} \tag{76}$$

$$\frac{\partial R}{\partial RP_i} = \frac{\partial R}{\partial r_i} \frac{\partial r_i}{\partial RP_i} \tag{77}$$

where $\frac{\partial r_i}{\partial RP_i}$ is interpolated off the reference trajectory at t_{obs}

$$\frac{\partial R}{\partial T_i} = \frac{\partial R}{\partial r_i} \frac{\partial r_i}{\partial T_i} \tag{78}$$

$$\frac{\partial r_i}{\partial T_i} = \begin{cases} 0 & if \, t_{obs} \le t_{T_S} \\ [\phi_{T_i}(t_{obs}) - \phi_{e_i}(t_{obs})\phi_{e_i}^{-1}(t_{T_S})\phi_{T_i}(t_{T_S})]_r & if \, t_{T_S} < t_{obs} \le t_{T_E} \\ [\phi_{e_i}(t_{obs})(\phi_{e_i}^{-1}(t_{T_E})\phi_{T_i}(t_{T_E}) - \phi_{e_i}^{-1}(t_{T_S})\phi_{T_i}(t_{T_S}))]_r & if \, t_{obs} > t_{T_E} \end{cases} \tag{79}$$

where ϕ_{e_i} = partials of position and velocity with respect to epoch orbital elements obtained by interpolating off the reference trajectory at the appropriate time

ϕ_{T_i} = partials of position and velocity with respect to the canonical thrust parameters obtained by interpolating off the reference trajectory at the appropriate time

and $[\ \ ]_r$ denotes the first three rows of the 6×3 matrix

$$\frac{\partial R}{\partial PM} = \frac{\partial R}{\partial r_i}\left(\frac{\partial r_i}{\partial p} \quad \frac{\partial r_i}{\partial q} \quad \frac{\partial r_i}{\partial \Delta \dot{t}(T_0)} - (t_0 - T_0)\frac{\partial r_i}{\partial \Delta t(T_0)}\right) + \frac{\partial R}{\partial r_k}\frac{\partial r_k}{\partial PM} \tag{80}$$

where $\frac{\partial r_i}{\partial p}$, $\frac{\partial r_i}{\partial q}$, $\frac{\partial r_i}{\partial \Delta \dot{t}(T_0)}$, and $\frac{\partial r_i}{\partial \Delta t(T_0)}$ are interpolated off the reference trajectory at t_{obs}, T_0 denotes the trajectory epoch, and

$$\frac{\partial r_k}{\partial PM} = (BCD)^T \begin{pmatrix} -z_{EF_k} & 0 & -\bar{\omega}(t_{obs} - t_0)y_{EF_k} \\ 0 & z_{EF_k} & \bar{\omega}(t_{obs} - t_0)x_{EF_k} \\ x_{EF_k} & -y_{EF_k} & 0 \end{pmatrix} \tag{81}$$

$$\frac{\partial R}{\partial GC} = \frac{\partial R}{\partial r_i}\frac{\partial r_i}{\partial GC} \tag{82}$$

where $\frac{\partial r_i}{\partial GC}$ is interpolated off the reference trajectory at t_{obs}

Observation equations and partial derivatives for all other data types are derived from the range observation equation and partial derivatives.

RANGE DIFFERENCE

Range difference observations can be either of two types: the result of differencing range observations at two different times or the result of integrating the Doppler-shifted frequency for a given time interval, Δt_{RD}. The latter type includes differencing accumulated delta ranges that are continuous-count, integrated Doppler from some epoch. Both types are treated the same in the measurement processing. These observations are pairwise correlated if the end of one range difference interval is the same as the beginning of the next interval. This correlation is accounted for in the processing as described in the FILTER ALGORITHM section. The nonlinear range difference (RD) observation equation is given by

$$\begin{aligned} RD_{i,k} = {} & |r_i(t_{obs}) - r_k(t_{obs})| - |r_i(t_{obs} - \Delta t_{RD}) - r_k(t_{obs} - \Delta t_{RD})| \\ & - \frac{c}{10^6}\Big[(\Delta t_i^0(t_{obs}) + \Delta t_i(t_{obs})) - (\Delta t_i^0(t_{obs} - \Delta t_{RD}) + \Delta t_i(t_{obs} - \Delta t_{RD}))\Big] \\ & + \frac{c}{10^6}\Big[(\Delta t_k^0(t_{obs}) + \Delta t_k(t_{obs})) - (\Delta t_k^0(t_{obs} - \Delta t_{RD}) + \Delta t_k(t_{obs} - \Delta t_{RD}))\Big] \\ & + \frac{\Delta C_{R_i}(t_{obs})}{\sin E_{i,k}(t_{obs})} - \frac{\Delta C_{R_i}(t_{obs} - \Delta t_{RD})}{\sin E_{i,k}(t_{obs} - \Delta t_{RD})} \end{aligned} \tag{83}$$

where t_{obs} = end time of the range difference interval

and Δt_{RD} = range difference interval in seconds.

This equation is just the range observation equation evaluated at two times and differenced. The computed value is the difference of two computed range values. Therefore, the partial derivatives are the range partial derivatives differenced, where the derivatives at both times are based on the same time t_j. As a result of this differencing, only the changes in the clocks over the interval Δt_{RD} are relevant, i.e., $\frac{\partial RD}{\partial \Delta t_i} = \frac{\partial RD}{\partial \Delta t_k} = 0.$

Differenced and doubly-differenced range difference data types are not included in the MSF/S system. However, range difference data can be processed in a mode that emulates these data types. The differenced range difference emulation is based on the idea that single differencing to remove the satellite clock frequency offsets from the data is equivalent to solving for an independent frequency offset for every group of simultaneous observations of that satellite. This can be emulated by setting the clock frequency offset state process noise variance to a large value to decorrelate the estimates between mini-batch steps. The satellite frequency drift state should not be solved for. For doubly differencing, this variance adjustment must also be done for all station clocks except the master clock. This processing technique fully accounts for the measurement correlations introduced by differencing. However, this emulation is not exact if more than one observation from a given satellite-station pair is present in the mini-batch interval.

DIFFERENCED RANGE

Differenced range (DR) observations are derived by differencing two simultaneous observations from the same satellite for any pair of stations. The purpose of this differencing is to eliminate the satellite clock from the observations. No account of the correlations introduced when two pairs of stations have one station's data in common is included in processing of these observations. The nonlinear differenced range observation equation is given by

$$
\begin{aligned}
DR_{i,(k,k')} &= |\mathbf{r}_i(t_{obs}) - \mathbf{r}_k(t_{obs})| - |\mathbf{r}_i(t_{obs}) - \mathbf{r}_{k'}(t_{obs})| \\
&+ \frac{c}{10^6}\left[(\Delta\iota_k^{\,0}(t_{obs}) + \Delta\iota_k(t_{obs})) - (\Delta\iota_{k'}^{\,0}(t_{obs}) + \Delta\iota_{k'}(t_{obs}))\right] \qquad (84) \\
&+ \frac{\Delta C_{R_k}(t_{obs})}{\sin E_{i,k}(t_{obs})} - \frac{\Delta C_{R_{k'}}(t_{obs})}{\sin E_{i,k'}(t_{obs})}
\end{aligned}
$$

where k and k' are indicies specifying the two stations. Again, the computed value is obtained by differencing two range computed values, and the partial derivatives are obtained by differencing the two range partial derivatives. Zeroes are used for the range partial derivatives involving the other station. This differencing results in the partials for satellite clock states being zero.

Range data can be processed in a mode that emulates this data type. This emulation is based on the idea that single differencing to remove the satellite clock time offset from the data is equivalent to solving for an independent satellite time offset for every group of simultaneous observations of that satellite. This can be emulated by setting the clock time offset state process noise variance to a large value to decorrelate the estimates between mini-batch steps. The satellite frequency offset and drift states should not be solved for. This processing technique fully accounts for the correlations introduced by differencing. However, this emulation is not exact if more than one observation from a given satellite-station pair is present in the mini-batch interval.

DOUBLY-DIFFERENCED RANGE

Doubly-differenced range (DDR) observations are derived by differencing two simultaneous differenced range observations from any pair of satellites for the same pair of stations. The purpose of this differencing is to eliminate the station clocks from the observations in addition to the satellite clocks that were eliminated by the first differencing. No account of the correlations introduced by this differencing technique is included in the processing of these observations. The nonlinear doubly-differenced range observation equation is given by

$$
\begin{aligned}
DDR_{(i,i'),(k,k')} &= (|\mathbf{r}_i(t_{obs}) - \mathbf{r}_k(t_{obs})| - |\mathbf{r}_i(t_{obs}) - \mathbf{r}_{k'}(t_{obs})|) \\
&- (|\mathbf{r}_{i'}(t_{obs}) - \mathbf{r}_k(t_{obs})| - |\mathbf{r}_{i'}(t_{obs}) - \mathbf{r}_{k'}(t_{obs})|) \\
&+ \Delta C_{R_k}(t_{obs})\left(\frac{1}{\sin E_{i,k}(t_{obs})} - \frac{1}{\sin E_{i',k}(t_{obs})}\right) \qquad (85) \\
&- \Delta C_{R_{k'}}(t_{obs})\left(\frac{1}{\sin E_{i,k'}(t_{obs})} - \frac{1}{\sin E_{i',k'}(t_{obs})}\right)
\end{aligned}
$$

where i, i' are indices specifying the two satellites and k, k' are the station indices. The computed value is obtained by differencing two differenced range computed values, and the partial derivatives are obtained by differencing the two differenced range partial derivative sets with appropriate zeroes for irrelevant parameters. This differencing results in all clock partials being zero.

Range data can be processed in a mode that emulates this data type. In addition to emulating differenced range data, all station clock time offset states (except the master clock) should have their process noise variances set to large values. Also, all station frequency offset states should not be solved for. The comments given under the discussion of emulating differenced range also apply to doubly-differenced range emulation.

FILTER/SMOOTHER PROCESSING FLOW

Figure 4 gives a functional definition of the processing flow within the MSF/S system of programs. The Filter is initialized at t_0 or restarted at an arbitrary mini-batch step if previous filtering has been done. At each mini-batch step a measurement update is performed first, followed by solving the equations and generating diagnostics if required, and then propagating to the next mini-batch step. No propagated solution is ever computed unless observations are not present in a given mini-batch interval and solutions at each mini-batch step are required. This process is repeated until the solution and diagnostics at t_N are completed. At this point the y parameter solutions are final, i.e., no smoothing of the y parameters is possible. If stochastic orbit-related parameters are not present, the resulting orbital element and constant force model parameter corrections can be applied to their initial values and used to reintegrate a trajectory if the orbit is not converged, or an improved trajectory can be linearly propagated as in a batch fit. The polar motion and station coordinate tables are updated at this point. The updated station coordinates are required for the residual generation procedures. The appropriate information arrays and partial derivative matrices must be saved from the Filter at each mini-batch step to be used in the smoothing process. A Filter propagated trajectory can be created at this time if required and also a set of improved initial conditions.

Two paths within the Smoother are possible. If state and covariance estimates are required the left side is followed. If state estimates only are required the right side is followed. In the first case the smoothing arrays are initialized at t_N and smoothing proceeds in reverse time order. At each mini-batch step smoothing arrays are manipulated followed by solving the equations and generating diagnostics. This is repeated at each step until the process terminates at t_0. Smoothed trajectories can then be propagated and/or the SATRACK covariance matrices can be computed. In the second case the state estimates are initialized at t_N as in the first case and smoothing proceeds in reverse time order. At each mini-batch step the state estimate is computed based on the estimate at the previous step, the information arrays saved in the Filter, and the stochastic state equations. This process is repeated until t_0 is reached. No covariance information can be computed for this option but the rest of the diagnostics can. Trajectories can then be propagated followed by generation of residuals.

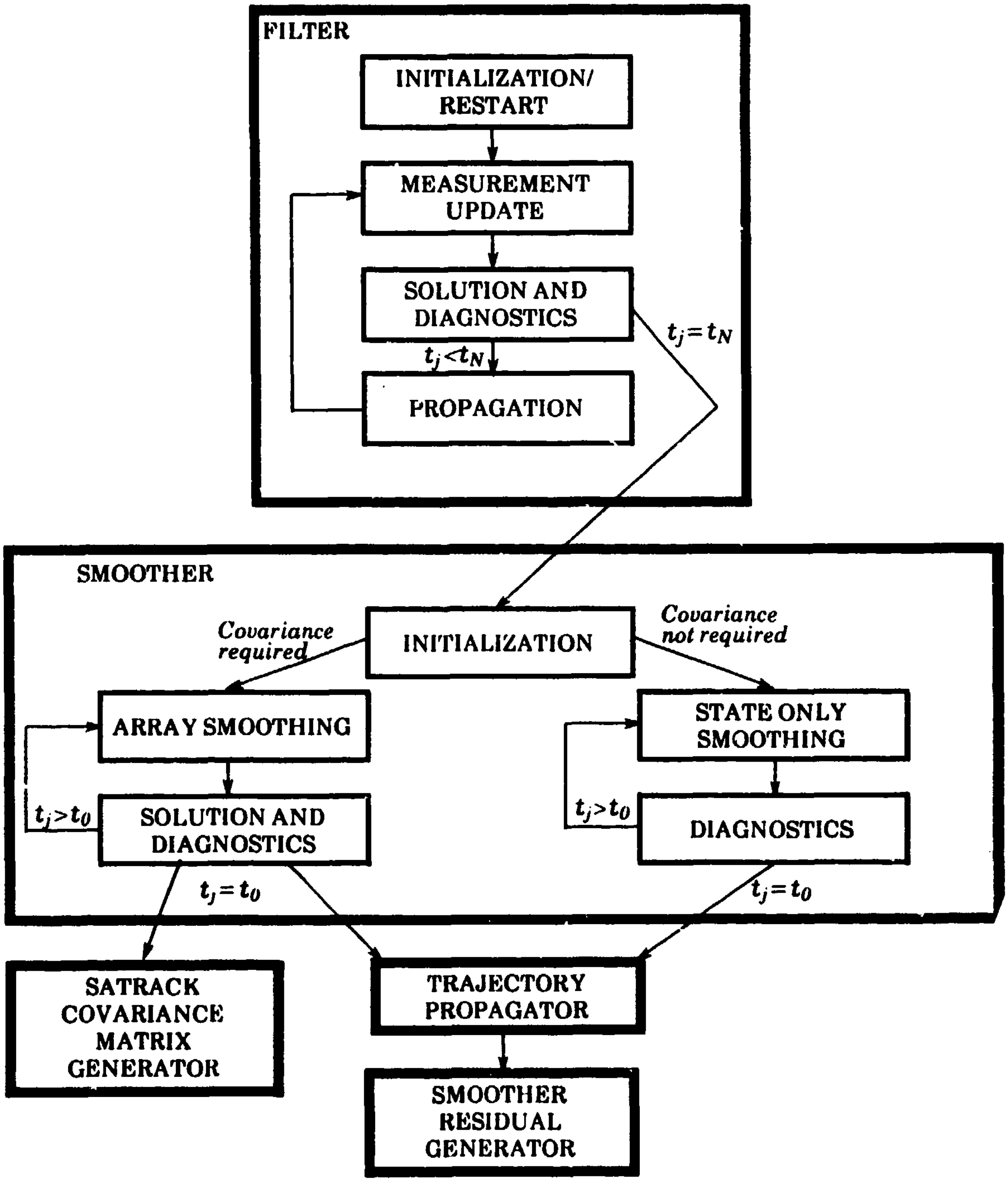


FIGURE 4. FILTER/SMOOTHER PROCESSING FLOW

The rest of this report will detail the square root information implementation of the Filter and Smoother algorithms, the method of obtaining the solution and diagnostics (these are identical in each algorithm except for handling the y parameters and for residual generation), the trajectory propagation procedures, and the SATRACK covariance matrix generation formulation. Derivations of some of the equations and showing that they are equivalent to the standard Kalman filter/RTS smoother equations are included in the appendicies. Also a discussion of properties of Householder transformations and the upper triangular matrix inversion method are given in the appendicies.

FILTER ALGORITHM

The Filter can be described as a square root information filter based on the form of the state equations given in the **STATE EQUATIONS** section and the observation equations given in the **OBSERVATION EQUATIONS AND PARTIAL DEVIRATIVES** section. The Filter consists of four processing steps, the last three of which are repeated: Initialization/Restart, Measurement Update, Solution and Diagnostics (optional), and Propagation. After initialization or restart of the information arrays, at each mini-batch step, t_j, a measurement update is performed if there are any observations in the corresponding mini-batch interval. Then the solution and diagnostics are computed if required and the information array is augmented and propagated from t_j to t_{j+1}. Portions of the information array required for smoothing are then saved. This cycle repeats until a measurement update at t_N has been done. The y parameter information at this time is final and is used throughout the smoothing procedure. Details for each processing step are given below.

INITIALIZATION/RESTART

The primary purpose of this step is to set up and initialize the information arrays. The parameters, if present, are always ordered as given in the state equation description, i.e., p parameters with orbit-related ones before measurement-related ones, x parameters, and then y parameters. In addition since both orbit-related p parameters and tropospheric refraction are modeled as first-order Gauss-Markov processes, these are at the top of the parameter list so that the propagation step computations can be made more efficient. The following notation is used in the rest of this report.

N_D = number of orbit-related p parameters, maximum of $6N_{SV}$

N_M = number of measurement-related p parameters, maximum of $3N_{SV} + 3N_{MS}$

N_p = number of p parameters (must always be > 0) $= N_D + N_M$

N_{GM} = number of Gauss-Markov p parameters, maximum of $6N_{SV} + N_{MS}$

N_x = number of x parameters, maximum of $6N_{SV}$

N_{y_s} = number of station-related y parameters, maximum of $3N_{MS}$

N_{y_f} = number of orbit-related y parameters, maximum of $3N_{SV} + 3N_T + N_{PM} + N_{GC}$

N_y = number of y parameters = $N_{y_s} + N_{y_f}$

N_{TOT} = total number of parameters = $N_p + N_x + N_y$

where N_{SV} = number of satellites

N_{MS} = number of stations

N_T = number of thrusts

N_{PM} = number of polar motion parameters = 3

N_{GC} = number of gravity coefficient parameters

The parameter set for a particular fit is selectable but with the following restrictions:

1. All orbit-related parameters except for thrusts, polar motion, and gravity coefficients are present for all satellites and individual parameters must be selectively deweighted if required.

2. All measurement-related parameters are present for all satellites or stations and individual parameters must be selectively deweighted if required. The clock parameters for the master station are automatically deweighted and the corresponding white noise spectral densities are set to approximately zero.

3. All station-related parameters are present for all stations and must be selectively deweighted if required. If orbits and station coordinates are being solved for simultaneously, the east component of one station should be deweighted.

4. If only differenced range and doubly-differenced range are being processed, satellite clock parameters are not included in the state equations. If only doubly-differenced data are being processed, satellite and station clock parameters are not included.

5. If only range difference data is being processed, all satellite and station time offset parameters should not be deweighted unless all the white noise spectral densities are zero.

6. Stochastic orbit-related parameters can only be solved for if pseudoepoch orbital elements are also present in the equations.

The following notation is used in the rest of this report:

~ indicates a Filter predicted quantity
^ indicates a Filter estimated quantity
* indicates a Smoother estimated quantity

The initial forms of the two information arrays at t_0 are given by

$$\underbrace{\begin{pmatrix} \tilde{R}_{p_0} & 0 & 0 & 0 \\ 0 & \tilde{R}_{x_0} & 0 & 0 \end{pmatrix}}_{N_{TOT}\qquad 1} \Big\} N_p + N_x \qquad \qquad \underbrace{(\tilde{R}_{y_0}}_{N_y} \;\; \underbrace{0)}_{1} \,\} \; N_y \tag{86}$$

p-x information array $\qquad$ y information array

where $\tilde{R}_{p_0}$ is an $N_p \times N_p$ diagonal matrix with each element of the form $1/\sigma_p$ and $\tilde{R}_{y_0}$ is an $N_y \times N_y$ diagonal matrix with each element of the form $1/\sigma_y$. σ_p and σ_y are *a priori* parameter sigmas in internal units. (The y array is not present if $N_y = 0$.) The $\tilde{R}_{x_0}$ matrix is $N_x \times N_x$ and has the form

$$\tilde{R}_{x_0} = \begin{pmatrix} \tilde{R}_{e_1} & & 0 \\ & \tilde{R}_{e_2} & \\ 0 & & \ddots \; \tilde{R}_{e_{N_{SV}}} \end{pmatrix} \tag{87}$$

where each $\tilde{R}_{e_i}$ is a 6×6 matrix computed for each satellite by

$$\tilde{R}_e = C_{RAC}^{-1} \bar{R}^T T' \tag{88}$$

$$\text{where } C_{RAC} = \begin{pmatrix} \sigma_R^2 & & & & & \\ & \sigma_A^2 & & & 0 & \\ & & \sigma_C^2 & & & \\ & & & \sigma_{\dot{R}}^2 & & \\ & 0 & & & \sigma_{\dot{A}}^2 & \\ & & & & & \sigma_{\dot{C}}^2 \end{pmatrix} \tag{89}$$

σ's are input *a priori* sigmas on radial, along-track, and cross-track position (in km) and velocity (in km/sec) at the fit epoch.

$$\bar{R} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \tag{90}$$

$$R = \left(\hat{r} \quad \frac{\hat{r} \times \hat{v}}{|\hat{r} \times \hat{v}|} \times \hat{r} \quad \frac{\hat{r} \times \hat{v}}{|\hat{r} \times \hat{v}|} \right) \tag{91}$$

$\mathbf{r}$, $\mathbf{v}$ are inertial coordinates of the satellite at the fit epoch interpolated off of the reference trajectory.

$$T' = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial e_s} & \cdots & \frac{\partial x}{\partial \Omega} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial e_s} & \cdots & \frac{\partial y}{\partial \Omega} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \dot{z}}{\partial a} & \frac{\partial \dot{z}}{\partial e_s} & \cdots & \frac{\partial \dot{z}}{\partial \Omega} \end{pmatrix} \tag{92}$$

T' is a matrix of partial derivatives of position and velocity at the fit epoch with respect to orbital elements at the trajectory epoch interpolated off of the reference trajectory.

$\tilde{R}_e$ is just the fit epoch RAC position and velocity sigmas transformed to a covariance matrix on orbital elements at the trajectory epoch, inverted and with the square root taken.

To restart the Filter the assumption is made that all physical constants, spectral densities, decorrelation times, and other quantities must be the same as those used in the last execution of the Filter. The two information arrays are then initialized with the results of the propagation step from $t_{\ell-1}$ to t_ℓ as follows:

p-x information array / y information array

$$\begin{pmatrix} \tilde{R}_p & \tilde{R}_{px} & \tilde{R}_{py} & \tilde{z}_p \\ \tilde{R}_{xp} & \tilde{R}_x & \tilde{R}_{xy} & \tilde{z}_x \end{pmatrix}_\ell \qquad (\hat{R}_y \quad \hat{z}_y)_{\ell-1} \tag{93}$$

MEASUREMENT UPDATE

All observations in the mini-batch interval $\left(t_j - \frac{\Delta t}{2}, t_j + \frac{\Delta t}{2} \right]$ are processed simultaneously because computationally this is the most efficient. Measurement updating is carried out by augmenting the propagated information arrays with the whitened and decorrelated measurement matrix and residuals (A z) and tranforming this expanded array using Householder orthogonal transformations. The details of these procedures are given as follows:

1. All observations with observation times, t_{obs}, such that $t_j - \frac{\Delta t}{2} < t_{obs} \leq t_j + \frac{\Delta t}{2}$ are processed for the stations and satellites selected unless:

   a. this data type is not being processed or t_{obs} lies outside of the subspan for this particular data type,
   b. the corresponding pass number is in the list of passes to be deleted,

c. the observation sigma indicates that this observation has been previously tagged (negative sigmas indicate this),
d. any elevation angle is less than $E_{tol.}$—an input tolerance in degrees, or
e. a clock event for the given station or satellite is present in this interval.

At the start time, $t_S^{Red.}$, of the reduced mini-batch step span the mini-batch interval is defined as $\left(t_S^{Red.} - \frac{\Delta t}{2}, t_S^{Red.} + \frac{\Delta t^{Red.}}{2}\right]$. At the end time, t_E^{Red}, of the reduced mini-batch step span the mini-batch interval is defined as $\left(t_E^{Red.} - \frac{\Delta t^{Red.}}{2}, t_E^{Red.} + \frac{\Delta t}{2}\right]$. If no observations are present in a given mini-batch interval, the Householder transformation is still done in order to upper triangularize the array, i.e., the transformation indicated in equation (103) below is done with the $(A \;\; z)$ rows absent. If no solution is required the processing skips to the propagation step.

2. For all remaining observations, the partial derivative matrices (designated A') relating the observations to each parameter are computed along with the residuals (designated $O\text{–}C'$) as described in the **OBSERVATION EQUATIONS AND PARTIAL DERIVATIVES** section. Only those partial derivatives with respect to the relevant satellite and station parameters are non-zero. All others are zero with each column corresponding to a parameter as ordered in the state equations. These quantities are denoted by

$$\begin{array}{ll} A' = (A'_p \;\; A'_x \;\; A'_y) & z' = O\text{–}C' \\ M_j \times (N_p + N_x + N_y) & M_j \times 1 \end{array} \tag{94}$$

where M_j = number of observations in the mini-batch interval centered at t_j.

3 All observations are then whitened and decorrelated. To whiten range, differenced range, and doubly-differenced range each row of the partial derivative matrix A' and the residuals z' is divided by its observation sigma, σ_{obs}, or by $\sigma_{obs_{MIN}}$ for that data type if $\sigma_{obs} < \sigma_{obs_{MIN}}$ to get A and z. For the range difference data where $t_{obs} - \Delta t_{RD} \neq t_{obs}$(previous), i.e., the beginning of the range difference interval is not identical to the observation time of the previous observation from this same station-satellite pair, a similar procedure is used. However, if the times do coincide the observations are correlated and must be whitened and decorrelated before being processed. The range difference data are whitened and decorrelated as follows (see Appendix F for the derivation of this procedure):

If the observation is not tagged the two input sigmas for the observations differenced to obtain the range difference observation, designated $\sigma_{ADR_{n-1}}$ and σ_{ADR_n}, are possibly redefined as follows:

$$\sigma_{ADR_{n-1}} = \frac{\sigma_{RD_{MIN}}}{\sqrt{2}} \text{ if } 0 < \sigma_{ADR_{n-1}} < \frac{\sigma_{RD_{MIN}}}{\sqrt{2}} \tag{95}$$

$$\sigma_{ADR_n} = \frac{\sigma_{RD_{MIN}}}{\sqrt{2}} \text{ if } \sigma_{ADR_n} < \frac{\sigma_{RD_{MIN}}}{\sqrt{2}} \tag{96}$$

The range difference observation sigma is then given by

$$\sigma_{RD_n} = (\sigma^2_{ADR_{n-1}} + \sigma^2_{ADR_n})^{\frac{1}{2}} \tag{97}$$

If $\sigma_{ADR_{n-1}} = 0$ or the last range difference observation for this satellite-station pair was not processed, the observation is whitened by dividing A'_{RD_n} and z'_{RD_n} by σ_{RD_n}. $\bar{\sigma}_n$ is then set equal to σ_{RD_n}, and $\bar{\sigma}_n$, A_{RD_n}, and z_{RD_n} are saved. If $\sigma_{ADR_{n-1}} \neq 0$ the observation is whitened and decorrelated as follows:

$$\bar{\rho}_{n-1} = \frac{-\sigma^2_{ADR_{n-1}}}{\bar{\sigma}_{n-1}} \tag{98}$$

$$\bar{\sigma}_n = (\sigma^2_{RD_n} - \bar{\rho}^2_{n-1})^{\frac{1}{2}} \tag{99}$$

$$A_{RD_n} = (A'_{RD_n} - \bar{\rho}_{n-1} A_{RD_{n-1}})/\bar{\sigma}_n \tag{100}$$

$$z_{RD_n} = (z'_{RD_n} - \bar{\rho}_{n-1} z_{RD_{n-1}})/\bar{\sigma}_n \tag{101}$$

$\bar{\sigma}_n$, A_{RD_n}, and z_{RD_n} are then saved. It is assumed that if an entire mini-batch interval is processed before the same satellite-station pair occurs again, the new observation is uncorrelated with the previous observation. This is done so that information does not have to be saved for more than one mini-batch interval.

The effect of the accumulated clock noise during the Δt_{RD} interval is not modeled so the minimum observation sigma for range difference data should be used to account for this effect.

4. The propagated p-x and y information arrays without the terms required for smoothing are given by

$$\underset{(N_p + N_x) \times (N_p + N_x + N_y + 1)}{\begin{pmatrix} \tilde{R}_p & \tilde{R}_{px} & \tilde{R}_{py} & \tilde{z}_p \\ \tilde{R}_{xp} & \tilde{R}_x & \tilde{R}_{xy} & \tilde{z}_x \end{pmatrix}_j} \qquad \underset{N_y \times (N_y + 1)}{(\tilde{R}_y \;\; \tilde{z}_y)_j = (\hat{R}_y \;\; \hat{z}_y)_{j-1}} \tag{102}$$

The measurement update is done in two steps if y parameters are present. The p-x information array is augmented by $(A \;\; z)$, and a sequence of Householder orthogonal transformations, $\hat{T}_{px}$, are applied to zero out elements below the diagonal of the first $N_p + N_x$ columns:

$$\hat{T}_{px} \begin{pmatrix} \tilde{R}_p & \tilde{R}_{px} & \tilde{R}_{py} & \tilde{z}_p \\ \tilde{R}_{xp} & \tilde{R}_x & \tilde{R}_{xy} & \tilde{z}_x \\ A_p & A_x & A_y & z \end{pmatrix}_j = \begin{pmatrix} \hat{R}_p & \hat{R}_{px} & \hat{R}_{py} & \hat{z}_p \\ 0 & \hat{R}_x & \hat{R}_{xy} & \hat{z}_x \\ 0 & 0 & \hat{A}_y & \hat{z} \end{pmatrix}_j \tag{103}$$

$$(N_p + N_x + M_j) \times (N_p + N_x + N_y + 1)$$

where $\hat{R}_p$ and $\hat{R}_x$ are upper triangular matrices. If no y parameters are present the columns with a y subscript are absent, $\hat{z} = e$, and the measurement update is complete. The e term is related to the sum-of-squares of weighted residuals. If y parameters are present, the measurement update is completed by augmenting the y information array with $(\hat{A}_y\ \hat{z})$ and applying another sequence of Householder orthogonal transformations, $\hat{T}_y$, to zero out elements below the diagonal of the first N_y columns:

$$\hat{T}_y \begin{pmatrix} \tilde{R}_y & \tilde{z}_y \\ \hat{A}_y & \hat{z} \end{pmatrix}_j = \begin{pmatrix} \hat{R}_y & \hat{z}_y \\ 0 & e \end{pmatrix}_j \tag{104}$$

$$(N_y + M_j) \times (N_y + 1)$$

where both $\tilde{R}_y$ and $\hat{R}_y$ are upper triangular matrices, i.e., the y information array is always upper triangular. $(\tilde{R}_y\ \tilde{z}_y)$ could be added as extra rows in equation (103) above but this is not done to save storing a large block of zeroes that would never change.

The transformation matrices $\hat{T}_{px}$ and $\hat{T}_y$ are not explicitly computed. Householder transformations can be carried out by operating on the columns of a matrix one at a time as follows:

Let R be an arbitrary $m \times n$ information array. To zero out all elements below the diagonal of the first column define a scalar s and a vector u by

$$s = -sgn(R(1,1)) \left(\sum_{i=1}^{m} [R(i,1)]^2 \right)^{1/2} \tag{105}$$

$$u(1) = R(1,1) - s \tag{106}$$

$$u(i) = R(i,1) \qquad i = 2,3,\ldots,m \tag{107}$$

$$\alpha = \frac{1}{s\,u(1)} \tag{108}$$

Then define $\gamma_j = \alpha \sum_{i=1}^{m} u(i)\, R(i,j)$ (109)

The effect of the transformation T_u on a column j is then written as

$$T_u(R(i,j)) = R(i,j) + \gamma_j u(i) \qquad i = 1,2\ldots,m \tag{110}$$

This results in the first column being $\begin{pmatrix} s \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ as shown in Appendix G. This is the intended result and this must be done to every column of the array. This procedure is then repeated to zero out the below diagonal elements of the second column with u defined by the last $m-1$ rows of this column. This procedure (in which the matrix operated on decreases in both dimensions at each step) is repeated until the required number of columns have been transformed. A property of Householder transformations that affected the implementation of equations (103) and (104) is discussed in Appendix G also.

5. If any diagnostics are to be computed or a trajectory is to be propagated, the solution is then computed as described in the **SOLUTION AND DIAGNOSTICS** section. If a master station switch has taken place in this interval the two diagonal elements of the $\hat{R}_p$ array corresponding to the new master station clock parameters are set to large values after solution. If the last mini-batch interval centered at t_N has been processed the filtering is complete and a solution is computed for the y parameters if present.

PROPAGATION

Propagation from t_j to t_{j+1} involves modifying the p-x information array to incorporate the effects of process noise. Bias parameters are unaffected by process noise so that $\tilde{R}_{y_{j+1}} = \hat{R}_{y_j}$. Propagation also involves generation of auxiliary arrays required for smoothing. The propagation step is based on the state transition and R_w (derived from Q) matrices described in the **STATE EQUATIONS** section. An augmented p-x information array is upper triangularized over the first N_p columns to carry out the propagation as follows:

$$\tilde{T}_p \begin{array}{c} \begin{array}{ccccc} \overbrace{\phantom{-\hat{R}_p-\hat{R}_{px}V_p}}^{N_p} & \overbrace{}^{N_p} & \overbrace{\phantom{\hat{R}_{px}}}^{N_x} & \overbrace{\phantom{\hat{R}_{py}}}^{N_y} & 1 \end{array} \\ \begin{pmatrix} -R_w M & R_w & 0 & 0 & 0 \\ \hat{R}_p - \hat{R}_{px} V_p & 0 & \hat{R}_{px} & \hat{R}_{py} & \hat{z}_p \\ -\hat{R}_x V_p & 0 & \hat{R}_x & \hat{R}_{xy} & \hat{z}_x \end{pmatrix}_j \end{array} = \tag{111}$$

$$\begin{array}{cl} \begin{pmatrix} R_p^* & R_{pp}^* & R_{px}^* & R_{py}^* & z_p^* \\ 0 & \tilde{R}_p & \tilde{R}_{px} & \tilde{R}_{py} & \tilde{z}_p \\ 0 & \tilde{R}_{xp} & \tilde{R}_x & \tilde{R}_{xy} & \tilde{z}_x \end{pmatrix}_{j+1} & \begin{array}{l} \}\ N_p \\ \}\ N_p \\ \}\ N_x \end{array} \end{array}$$

where R_p^* is an upper triangular matrix and * denotes the smoother-related matrices required to obtain smoothed state and covariance estimates at t_j. The Househoulder transformations are done as described above in the Measurement Update processing step except that a computational reduction is made. All columns of the $-R_w M$ matrix have zeroes below the diagonals down to and including row N_p. When applying the transformation to the first column, rows 2 through N_p for all columns do not change because of a property of Householder transformations given in Appendix G. These rows are ignored in the computations to save on computer time. This reduction applies to the first $N_p - 1$ columns as they are zeroed out below the diagonal. Another property of Householder transformations given in Appendix G is also applied in implementing equation (111) and is possible because R_w is diagonal for the first N_{GM} rows and columns. When processing the first column the only additional column that changes out of the the first $2\,N_p$ columns is the $N_p + 1$ column. When processing the second column only the $N_p + 1$ and $N_p + 2$ columns change out of the first $2\,N_p$ columns. One additional column is affected until column $N_{GM} + 1$ is processed. The part of R_p^* corresponding to the Gauss-Markov parameters is upper triangular and the part of R_{pp}^* corresponding to these same parameters is lower triangular. This is the reason why the Gauss-Markov parameters are ordered first in the state equations. The R_w and M arrays are a function of the mini-batch step size so they are adjusted starting at $t_S^{Red.}$ and then reset to their original values at $t_E^{Red.}$. Portions of the R_w matrices corresponding to the satellite and station clocks are adjusted appropriately for events as required. The predicted solution is never computed in a square root information filter because either a full matrix inversion or a Householder transformation followed by inversion of an upper triangular matrix is required.

Appendix H discusses the derivation of the measurement update equations (103) and (104) and the propagation equations (111). Also this appendix shows the mathematical equivalence between these equations (actually a more general form of the propagation equations) and the Kalman filter equations given by equations (3) thru (7) in the **ESTIMATION CONCEPTS** section.

SMOOTHER ALGORITHMS

Two smoother algorithms are included. The first uses Householder transformations to upper triangularize smoothing information arrays from which both state and covariance estimates can be computed. This is referred to as the Array Smoother here but is commonly called the square root information smoother. The second combines the stochastic state equations and data equations derived from the information arrays saved in the propagation step of the Filter to obtain state estimates only, i.e., no covariance information is available. This is referred to as the State Only Smoother. The Array Smoother consists of three processing steps, the last two of which are repeated: Initialization, Array Smoothing, and Solution and Diagnostics. At each mini-batch step t_j a smoothing array is constructed using the array solved at t_{j+1} and the array saved in the Filter when propagating from t_j to t_{j+1}. This array is upper triangularized for $2N_p + N_x$ columns and combined with the y parameter solution matrices from t_N (which remain constant throughout the span) to obtain the state and covariance estimates. This cycle repeats until the solution at t_0 has been computed. Details for the first two processing steps are given below followed by a description of the State Only Smoother.

INITIALIZATION

The purpose of this step is to set up and initialize the information arrays. This step is common to the two smoother algorithms. The order of the parameters is the same as that used in the Filter. At t_N the arrays are initialized with the arrays determined after the measurement update at t_N done in the Filter. The Smoother solution at t_N is identical to the Filter solution at t_N. The information arrays at t_N are given by

$$\text{p-x information array} \qquad \begin{array}{c} \begin{matrix} \underbrace{}_{N_p} & \underbrace{}_{N_x} & \underbrace{}_{N_y} & 1 \end{matrix} \\ \begin{pmatrix} \hat{R}_p & \hat{R}_{px} & \hat{R}_{py} & \hat{z}_p \\ 0 & \hat{R}_x & \hat{R}_{xy} & \hat{z}_x \end{pmatrix}_N \begin{matrix} \}\, N_p \\ \}\, N_x \end{matrix} \end{array} \qquad \text{y information array} \qquad \begin{array}{c} \begin{matrix} \underbrace{}_{N_y} & 1 \end{matrix} \\ (\hat{R}_y \quad \hat{z}_y)_N \;\}\, N_y \end{array} \tag{112}$$

where $\hat{R}_p$, $\hat{R}_x$, and $\hat{R}_y$ are upper triangular matrices. The y information array does not change in the smoothing process.

ARRAY SMOOTHING

Smoothing of the p-x information array at time t_j is accomplished by applying a sequence of Householder transformations to zero out elements below the diagonal of the first $2N_p + N_x$ columns of an augmented p-x information array as follows:

$$T^*_{ppx} \begin{pmatrix} \overbrace{\tilde{R}_{pp}(t_j)}^{N_p} & \overbrace{\tilde{R}_p(t_j) + \tilde{R}_{pp}(t_j)M + \tilde{R}_{px}(t_j)V_{p_j}}^{N_p} & \overbrace{\tilde{R}_{px}(t_j)}^{N_x} & \overbrace{\tilde{R}_{py}(t_j)}^{N_y} & \overbrace{\tilde{z}_p(t_j)}^{1} \\ R^*_p(t_{j+1}) & R^*_p(t_{j+1})M + R^*_{px}(t_{j+1})V_{p_j} & R^*_{px}(t_{j+1}) & R^*_{py}(t_{j+1}) & z^*_p(t_{j+1}) \\ 0 & R^*_x(t_{j+1})V_{p_j} & R^*_x(t_{j+1}) & R^*_{xy}(t_{j+1}) & z^*_x(t_{j+1}) \end{pmatrix} = \tag{113}$$

$$\begin{pmatrix} R'_{pp}(t_j) & R'_p(t_j) & R'_{px}(t_j) & R'_{py}(t_j) & z'_p(t_j) \\ 0 & R^*_p(t_j) & R^*_{px}(t_j) & R^*_{py}(t_j) & z^*_p(t_j) \\ 0 & 0 & R^*_x(t_j) & R^*_{xy}(t_j) & z^*_x(t_j) \end{pmatrix} \begin{matrix} \}\, N_p \\ \}\, N_p \\ \}\, N_x \end{matrix}$$

where the ~ terms correspond to the * terms saved in propagating in the Filter from t_j to t_{j+1}. $\tilde{R}_p$, R'_{pp}, R^*_p, and R^*_x are all upper triangular matrices. The Householder transformation is carried out as described above in the Measurement Update processing step. These computations also take advantage of the sparseness of the V_{p_j} matrix and the fact that M is diagonal. The terms identified with the ' superscript are not required for any further computations. The solution method for this approach is given in the SOLUTION AND DIAGNOSTICS section. Appendix I contains a derivation of these smoothing equations and also shows the mathematical equivalence between these equations (actually a more general form) and the RTS smoother equations given by equations (8) thru (10) in the ESTIMATION CONCEPTS section.

STATE ONLY SMOOTHING

This approach utilizes the stochastic state equations involving the p and x parameters along with the data equations derived from the information arrays saved in the Filter propagation step to recursively generate smoothed estimates for the p and x parameters. The initialization arrays given above are solved at t_N to get Δp_N^*, Δx_N^*, and Δy_N^*. Then given the smoothed state solution at t_{j+1}, the smoothed state solution at t_j is just given by

$$\Delta p_j^* = [\tilde{R}_p(t_j)]^{-1} \; [\tilde{z}_p(t_j) - \tilde{R}_{pp}(t_j)\Delta p_{j+1}^* - \tilde{R}_{px}(t_j)\Delta x_{j+1}^* - \tilde{R}_{py}(t_j)\Delta y_N^*] \tag{114}$$

$$\Delta x_j^* = \Delta x_{j+1}^* - V_{p_j} \Delta p_j^* \tag{115}$$

where ~ denotes matrices saved in propagating from t_j to t_{j+1} and $\tilde{R}_p(t_j)$ is always upper triangular. All elements of V_{p_j} which multiply non-orbit-related p parameters are zero and orbit-related p parameters for a given satellite affect the x parameter solution for that satellite only. This procedure requires the inverse of an $N_p \times N_p$ upper triangular matrix at each mini-batch step instead of an $(N_p + N_x) \times (N_p + N_x)$ matrix as in the Array Smoother algorithm.

SOLUTION AND DIAGNOSTICS

The solution method and the diagnostic computations are almost identical between the Filter and Array Smoother and are therefore described here together. The state solutions and all diagnostics depend on first computing the inverses of the upper triangular p-x and y information arrays. These inverses are always computed in the Array Smoother but are only computed in the Filter if any diagnostics are required or a trajectory is to be propagated. The possible diagnostics are correlation coefficients (at every nth mini-batch step), transformed corrections and standard deviations, total clock offsets, and residuals and signal-to-noise after fit. For the State Only Smoother all diagnostics are available except for the standard deviations and correlation coefficients.

SOLUTION

The solution (state), ΔX_j, and covariance, P_j, estimates are computed as follows:

$$\Delta X_j = \begin{pmatrix} \Delta p \\ \Delta x \\ \Delta y \end{pmatrix}_j = \begin{pmatrix} \begin{pmatrix} R_p & R_{px} \\ 0 & R_x \end{pmatrix}^{-1} \left[\begin{pmatrix} z_p \\ z_x \end{pmatrix} - \begin{pmatrix} R_{py} \\ R_{xy} \end{pmatrix} R_y^{-1} z_y \right] \\ R_y^{-1} z_y \end{pmatrix}_j \tag{116}$$

$$P_j = R_j^{-1} R_j^{-T} \tag{117}$$

$$R_j^{-1} = \begin{pmatrix} \begin{pmatrix} R_p & R_{px} \\ 0 & R_x \end{pmatrix}^{-1} & - \begin{pmatrix} R_p & R_{px} \\ 0 & R_x \end{pmatrix}^{-1} \begin{pmatrix} R_{py} \\ R_{xy} \end{pmatrix} R_y^{-1} \\ 0 & R_y^{-1} \end{pmatrix}_j \tag{118}$$

where all quantities denoted $\hat{}$ in the Filter and $*$ in the Array Smoother are evaluated at t_j except for the Smoother, in which R_y^{-1} and z_y are fixed at their values at t_N. All R_p, R_x, and R_y matrices are always upper triangular so that both $\begin{pmatrix} R_p & R_{px} \\ 0 & R_x \end{pmatrix}^{-1}$ and R_y^{-1} are upper triangular also, since upper traingularity is preserved by inversion. The inversion of upper triangular matrices is discussed in Appendix J. The solution is always computed for the y parameters at t_N in the Filter and may be computed at each mini-batch step if the y parameter only or full diagnostics are being computed. The solution information is saved in the Filter only if a trajectory is to be later propagated. In the Array Smoother this information is always saved because it is required for trajectory propagation and residual generation. The full covariance matrix is computed only for mini-batch steps for which the correlation coefficient matrix is required. Certain submatrices of the covariance matrix are required for deriving the standard deviations on transformed corrections.

At the last mini-batch time t_N in the Filter the final y parameter solutions are available. Each set of coordinates in the station coordinate table is then updated as follows, if these parameters were improved:

$$\begin{pmatrix} \lambda \\ \phi \\ h \end{pmatrix}_{Updated} = \begin{pmatrix} \lambda \\ \phi \\ h \end{pmatrix}_{Orig.} + \begin{pmatrix} \dfrac{180}{\pi} \dfrac{(1-e^2\sin^2\phi)^{1/2}\,\Delta E_N}{a_{Earth}\cos\phi} \\ \dfrac{180}{\pi} \dfrac{(1-e^2\sin^2\phi)^{3/2}\,\Delta N_N}{a_{Earth}(1-e^2)} \\ \Delta V_N \end{pmatrix} \tag{119}$$

$$\begin{pmatrix} x_{EF} \\ y_{EF} \\ z_{EF} \end{pmatrix}_{Updated} = \begin{pmatrix} x_{EF} \\ y_{EF} \\ z_{EF} \end{pmatrix}_{Orig.} + T \begin{pmatrix} \Delta E \\ \Delta N \\ \Delta V \end{pmatrix}_N \tag{120}$$

where $\begin{pmatrix} \Delta E \\ \Delta N \\ \Delta V \end{pmatrix}_N$ are the coordinate corrections at t_N and T is the transformation matrix defined in the **OBSERVATION EQUATIONS AND PARTIAL DERIVATIVES** section– equations (72)–(76). Each daily entry in the polar motion table is also updated as follows:

$$\begin{pmatrix} p \\ q \\ \Delta t \end{pmatrix}_{Updated} = \begin{pmatrix} p \\ q \\ \Delta t \end{pmatrix}_{Orig.} + \begin{pmatrix} \Delta p_N \\ \Delta q_N \\ \Delta(\dot{\Delta t})_N (t - t_0) \end{pmatrix} \tag{121}$$

where $t - t_0$ is in seconds.

Improved initial conditions at the trajectory epoch T_0 can be computed in the Filter if no stochastic orbit-related parameters are present. These improved initial conditions would primarily be used to reintegrate a trajectory if convergence has not occurred when emulating a batch fit processor. The required equations are given as follows:

$$\begin{pmatrix} r \\ \\ \dot{r} \end{pmatrix}_{Improved} = \begin{pmatrix} r \\ \\ \dot{r} \end{pmatrix}_{Ref.} + \begin{pmatrix} \dfrac{\partial r(T_0)}{\partial e} \\ \\ \dfrac{\partial \dot{r}(T_0)}{\partial e} \end{pmatrix} \Delta e_N \tag{122}$$

where the partial derivatives of position and velocity with respect to orbital elements are obtained by interpolating off of the trajectory at T_0.

$$K_{R_{Improved}} = K_{R_{Nom.}} + \Delta RP_N \tag{123}$$

$$T_{i_{Improved}} = T_{i_{Nom.}} + \Delta T_{i_N} \qquad i = 1,2 \tag{124}$$

CORRELATION COEFFICIENTS

The correlation coefficient matrix is computed every nth mini-batch step in terms of the actual solution parameters and not a transformed set. Each element of the covariance matrix is computed as follows:

$$p_{m,n} = p_{n,m} = \sum_{\ell = max(m,n)}^{N_{TOT}} r_{n,\ell} r_{m,\ell} \qquad n,m = 1,2,\ldots,N_{TOT} \tag{125}$$

where $r \varepsilon R^{-1}$ and $p \varepsilon P = R^{-1} R^{-T}$

and $\ell = max(m,n)$ takes advantage of the upper triangular form of R^{-1}. Each correlation coefficient is then computed as follows:

$$c_{m,n} = c_{n,m} = \begin{cases} 1. & \text{if } m = n \\ \dfrac{p_{m,n}}{\sqrt{p_{m,m}\, p_{n,n}}} & \text{if } m \neq n \end{cases} \qquad n,m = 1,2\ldots,N_{TOT} \tag{126}$$

Only a lower triangular array of correlation coefficients is computed since the covariance matrix is symmetric. The correlations between any pair of y parameters in the Smoother is the same at every mini-batch step.

TRANSFORMED CORRECTIONS AND STANDARD DEVIATIONS

All solution states, as mentioned before, are corrections to nominal values in internal units. On option these corrections and their corresponding covariances are converted to more meaningful corrections and standard deviations before being printed or plotted. If the state only smoothing option is selected no standard deviations can be computed. y parameter only corrections and standard deviations can also be computed in the Filter.

Stochastic radiation pressure parameter and all orbit-related y parameter corrections and standard deviations are unchanged. Gravitational acceleration parameter corrections and standard deviations are converted to 10^{-12}km/sec^2 for plotting so that they are in the same units as the y-axis acceleration parameter. Tropospheric refraction parameter corrections and standard deviations are converted to cm for plotting. Satellite clock parameter corrections and covariances are converted from pseudoepoch state to current state representations as follows:

$$\begin{pmatrix} \Delta\ddot{\iota}(t_j) \\ \Delta\dot{\iota}(t_j) \\ \Delta\iota(t_j) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t_j - t_0 & 1 & 0 \\ \dfrac{(t_j - t_0)^2}{2} & t_j - t_0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\ddot{\iota}_0 \\ \Delta\dot{\iota}_0 \\ \Delta\iota_0 \end{pmatrix}_j = \Phi_{C_{SV}} \Delta C_{SV}(t_j) \tag{127}$$

$$P_{C_{SV}}(t_j) = \Phi_{C_{SV}} \; P_{0_{C_{SV}}}(t_j) \, \Phi^T_{C_{SV}} \tag{128}$$

Frequency drift terms are converted from ppm/sec to parts in 10^{12}/day, frequency offset terms are converted from ppm to parts in 10^{12}, and time offset terms are converted from µsec to nsec. Station clock parameter corrections and covariances are also converted from pseudoepoch state to current state as follows:

$$\begin{pmatrix} \Delta\dot{\iota}(t_j) \\ \Delta\iota(t_j) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t_j - t_0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\dot{\iota}_0 \\ \Delta\iota_0 \end{pmatrix}_j = \Phi_{C_{MS}} \Delta C_{MS}(t_j) \tag{129}$$

$$P_{C_{MS}}(t_j) = \Phi_{C_{MS}} \; P_{0_{C_{MS}}}(t_j) \, \Phi^T_{C_{MS}} \tag{130}$$

Frequency offset terms are converted from ppm to parts in 10^{12} and time offset terms are converted from µsec to nsec. Only the square roots of the diagonals of $P_{C_{SV}}$ and $P_{C_{MS}}$ are required.

For each satellite the pseudoepoch state orbital element, radiation pressure, thrust(s), and gravity coefficient corrections and covariances are converted to position and velocity corrections and covariances in the RAC reference frame as follows:

$$\begin{pmatrix} \Delta \boldsymbol{r} \\ \Delta \dot{\boldsymbol{r}} \end{pmatrix}(t_j) = \bar{R}^T \left(\phi_e \Delta \boldsymbol{e}_j + \phi_{RP} \Delta \boldsymbol{RP}_j + \phi_{T_1} \Delta T_{1_j} + \phi_{T_2} \Delta T_{2_j} + \phi_{PM} \Delta \boldsymbol{PM}_j + \phi_{GC} \Delta \boldsymbol{GC}_j \right) \qquad (131)$$

$$\begin{aligned} P_{RAC}(t_j) = {} & \bar{R}^T \Big(\phi_e P_e \phi_e^T + \phi_{RP} P_{RP} \phi_{RP}^T + \phi_{T_1} P_{T_1} \phi_{T_1}^T + \phi_{T_2} P_{T_2} \phi_{T_2}^T + \phi_{PM} P_{PM} \phi_{PM}^T \\ & + \phi_{GC} P_{GC} \phi_{GC}^T + 2\phi_e P_{e,RP} \phi_{RP}^T + 2\phi_e P_{e,T_1} \phi_{T_1}^T + 2\phi_e P_{e,T_2} \phi_{T_2}^T + 2\phi_e P_{e,PM} \phi_{PM}^T \\ & + 2\phi_e P_{e,GC} \phi_{GC}^T \Big) \bar{R} \end{aligned} \qquad (132)$$

$$\text{where } \bar{R} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \qquad (133)$$

$$R = \left(\hat{\boldsymbol{r}} \quad \frac{\hat{\boldsymbol{r}} \times \hat{\boldsymbol{v}}}{|\hat{\boldsymbol{r}} \times \hat{\boldsymbol{v}}|} \times \hat{\boldsymbol{r}} \quad \frac{\hat{\boldsymbol{r}} \times \hat{\boldsymbol{v}}}{|\hat{\boldsymbol{r}} \times \hat{\boldsymbol{v}}|} \right) \quad \hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} \quad \hat{\boldsymbol{v}} = \frac{\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|} \qquad (134)$$

$\boldsymbol{r}, \dot{\boldsymbol{r}}$ = inertial position and velocity obtained by interpolating off of the trajectory at t_j

ϕ_e = 6×6 state transition matrix $\frac{\partial \boldsymbol{r}, \dot{\boldsymbol{r}}(t_j)}{\partial \boldsymbol{e}}$ obtained by interpolating off of the trajectory at t_j

$$\phi_{RP} = \begin{pmatrix} \frac{\partial \boldsymbol{r}(t_j)}{\partial \boldsymbol{RP}} \\ \frac{\partial \dot{\boldsymbol{r}}(t_j)}{\partial \boldsymbol{RP}} \end{pmatrix}$$ = partials of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ at t_j with respect to $\boldsymbol{RP}$ interpolated off of the trajectory

$$\phi_T = \begin{pmatrix} \frac{\partial \boldsymbol{r}(t_j)}{\partial \boldsymbol{T}} \\ \frac{\partial \dot{\boldsymbol{r}}(t_j)}{\partial \boldsymbol{T}} \end{pmatrix}$$ = partials of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ at t_j with respect to $\boldsymbol{T}$ computed using equation (79) in the <u>Range Partial Derivatives</u> subsection except evaluated at t_j and all 6 rows are required

$$\phi_{PM} = \begin{pmatrix} \frac{\partial \boldsymbol{r}(t_j)}{\partial \boldsymbol{PM}} \\ \frac{\partial \dot{\boldsymbol{r}}(t_j)}{\partial \boldsymbol{PM}} \end{pmatrix}$$ = partials of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ at t_j with respect to $\boldsymbol{PM}$ computed as in equation (80) in the <u>Range Partial Derivatives</u> subsection except evaluated at t_j and $\dot{\boldsymbol{r}}$ replaces $\boldsymbol{r}$ in computing the velocity partials

$$\Phi_{GC} = \begin{pmatrix} \dfrac{\partial r(t_j)}{\partial GC} \\ \dfrac{\partial \dot{r}(t_j)}{\partial GC} \end{pmatrix} = \text{partials of } r \text{ and } \dot{r} \text{ at } t_j \text{ with respect to } GC \text{ interpolated off of the trajectory}$$

P_e, P_{RP}, P_T, P_{PM}, P_{GC}, $P_{e,RP}$, $P_{e,T}$, $P_{e,PM}$, and $P_{e,GC}$ are portions of the full covariance matrix computed by equation (125) above. Only the square roots of the diagonal elements of P_{RAC} are required in km and km/sec. These are converted to meters and mm/sec for plotting.

The corrections and standard deviations for the two pole coordinates are converted to km at the Earth's surface by multiplying by a_{Earth} and then to meters for plotting. The correction and standard deviation for Δt are converted to msec/day.

TOTAL CLOCK OFFSETS

The current state satellite time and frequency offsets from GPS time and frequency are computed by adding together the nominal clock offsets and the current state solved-for clock correction at t_j, i.e.,

$$\Delta\iota_{Total}(t_j) = \Delta\iota^0(t_j) + \Delta\iota(t_j) \tag{135}$$

$$\Delta\dot{\iota}_{Total}(t_j) = \Delta\dot{\iota}^0(t_j) + \Delta\dot{\iota}(t_j) \tag{135}$$

The nominal clock may contain jumps and is computed in the Filter at each mini-batch step. The total satellite clock offsets can be computed in both the Filter and Smoother. The SATRACK project requires the Smoother-derived total clock offsets for both the satellites and stations. The values for the stations are obtained using parallel computations.

RESIDUALS AND SIGNAL-TO-NOISE

Residuals after fit are computed in the Filter for each mini-batch interval by linear adjustment of the original residuals, i.e.,

$$(O-C)_{adj.} = (O-C)' - A'\Delta X_j \tag{137}$$

where ΔX_j is the solution for all states and A' and $(O-C)'$ were saved before being whitened and decorrelated. The signal-to-noise ratio is defined as the square root of the weighted sum-of-squares of residuals divided by the number of observations. For each mini-batch interval it is computed after fit by

$$(S/N)_j(O-C) = \left[\frac{\sum_{m=1}^{M_j} \left(\dfrac{(O-C)_{adj.}}{\sigma_{obs}}\right)_m^2}{M_j} \right]^{1/2} \tag{138}$$

where σ_{obs} is the actual sigma used (may be different from the input value because of the minimum sigma override) and M_j is the total number of observations processed in the jth mini-batch interval. The signal-to-noise ratio for each mini-batch interval in the Filter is also available without computing adjusted residuals as a by-product of the square root information implementation. The e vector in the information array after measurement update is related to the signal-to-noise as follows:

$$(S/N)_j(\text{Filter}) = \left(\frac{\sum_{m=1}^{M_j} e_m^2}{M_j} \right)^{\frac{1}{2}} \tag{139}$$

Residuals and signal-to-noise after fit from the Smoother for each mini-batch interval are obtained by reprocessing the same observations used in the Filter. This is done by evaluating the observation equations using both the nominal and solved-for clock information, tropospheric refraction corrections, updated station coordinates, propagated trajectories (described in the next section), and updated polar motion information. If the corresponding Filter execution used previously computed total clock offsets for the satellites, this same information is used in generating residuals from the Smoother. Equation (138) is used with $(O-C)_{adj.}$ replaced by $(O-C)_{Smr.}$ to get the Smoother signal-to-noise ratio $(S/N)_j$(Smoother) for each mini-batch interval. Then the overall Smoother signal-to-noise ratio is computed by

$$\text{Overall } S/N \text{ (Smoother)} = \left[\frac{\sum_{j=0}^{N} M_j[(S/N)_j(\text{Smoother})]^2}{\sum_{j=0}^{N} M_j} \right]^{\frac{1}{2}} \tag{140}$$

where $N+1$ is the number of mini-batch steps (corresponds to t_0 thru t_N). The number, mean, standard deviation, and RMS of residuals for the entire fit span by satellite, station, and overall for each data type are computed. Residuals are converted from kilometers to meters for plotting. Residuals are also computed for observations not processed in the Filter because they were tagged, did not pass the elevation angle tolerance test, or were deleted by pass number. On option residuals for range difference, differenced range, and doubly-differenced range data can also be computed if range data for the same stations were processed in the Filter.

TRAJECTORY PROPAGATION

All orbit-related parameter corrections are transformed into inertial position and velocity corrections at each trajectory timeline using linear propagation techniques. These corrections are added to the reference trajectory positions and velocities at each timeline to produce the propagated trajectory. Earth-fixed position and velocity are obtained by transforming these improved inertial coordinates using improved values for polar motion if also solved for. The propagated trajectory span normally corresponds to the fit span with at least 4 extra timelines added on both ends to accomodate the interpolation method. However, in the case of no orbit-related p parameters being solved for ("batch" mode), the propagated trajectory is identical in length to the reference trajectory (except it has fewer extra timelines at each end) even if the fit span is a subset of the trajectory span. Also in this case the partial derivatives and other quantities (primarily solar radiation pressure model related items) on the original reference trajectory can be copied to the propagated trajectory if required. All satellites are processed simultaneously. Improved initial conditions required for integrating new trajectories can be computed after propagation is completed as described at the end of this section.

The corrections to inertial position and velocity at a given trajectory timeline T_ℓ are computed as follows:

$$\Delta r(T_\ell) = \frac{\partial r(T_\ell)}{\partial K_R(t_j)} \Delta K_{R_j} + \frac{\partial r(T_\ell)}{\partial G(t_j)} \Delta G_j + \frac{\partial r(T_\ell)}{\partial e} \Delta e_j \tag{141}$$

$$+ \frac{\partial r(T_\ell)}{\partial RP} \Delta RP_j + \frac{\partial r(T_\ell)}{\partial T_1} \Delta T_{1_j} + \frac{\partial r(T_\ell)}{\partial T_2} \Delta T_{2_j} + \frac{\partial r(T_\ell)}{\partial PM} \Delta PM_j + \frac{\partial r(T_\ell)}{\partial GC} \Delta GC_j$$

$$\Delta \dot{r}(T_\ell) = \frac{\partial \dot{r}(T_\ell)}{\partial K_R(t_j)} \Delta K_{R_j} + \frac{\partial \dot{r}(T_\ell)}{\partial G(t_j)} \Delta G_j + \frac{\partial \dot{r}(T_\ell)}{\partial e} \Delta e_j \tag{142}$$

$$+ \frac{\partial \dot{r}(T_\ell)}{\partial RP} \Delta RP_j + \frac{\partial \dot{r}(T_\ell)}{\partial T_1} \Delta T_{1_j} + \frac{\partial \dot{r}(T_\ell)}{\partial T_2} \Delta T_{2_j} + \frac{\partial \dot{r}(T_\ell)}{\partial PM} \Delta PM_j + \frac{\partial \dot{r}(T_\ell)}{\partial GC} \Delta GC_j$$

where the orbit-related parameter corrections at t_0 are used if $T_\ell < t_0$, the corrections at t_j are used if $t_j \le T_\ell < t_{j+1}$, and the corrections at t_N are used if $T_\ell \ge t_N$. The partial derivatives of position and velocity with respect to orbital elements, radiation pressure, thrust(s), polar motion, and gravity coefficients are the same as those defined in the **SOLUTION AND DIAGNOSTICS** section except they are evaluated at T_ℓ. The partial derivatives of position and velocity with respect to stochastic radiation pressure and gravitational acceleration parameters are the same as those defined in the **STATE EQUATIONS** section with t_{j+1} replaced by T_ℓ (equations (22) and (23) for K_R and equations (29) and (30) for G). These partials are zero if $T_\ell = t_j$. This is a result of the pseudoepoch state formulation of the equations. If a Filter "batch" mode propagated trajectory is being created the orbital element and orbit-related y parameter corrections at t_N are used at all trajectory timelines. If a Smoother propagated trajectory is being created the orbit-related y parameter corrections at t_N are also used since no smoothing of these corrections is possible.

The improved inertial position and velocity at a given trajectory timeline are then given by

$$r_{Improved}(T_\ell) = r_{Ref.}(T_\ell) + \Delta r(T_\ell) \tag{143}$$

$$\dot{r}_{Improved}(T_\ell) = \dot{r}_{Ref.}(T_\ell) + \Delta \dot{r}(T_\ell) \tag{144}$$

The improved Earth-fixed position and velocity are then given by

$$r_{EF}(T_\ell) = ABCD(T_\ell)r_{Improved}(T_\ell) \tag{145}$$

$$\dot{r}_{EF}(T_\ell) = ABCD(T_\ell)\dot{r}_{Improved}(T_\ell) + AWBCD(T_\ell)r_{Improved}(T_\ell) \tag{146}$$

where $$W = \begin{pmatrix} 0 & \tilde{\omega} & 0 \\ -\tilde{\omega} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{147}$$

When propagating a trajectory based on the Filter corrections the improved polar motion information is used. For p and q the corrections at t_j are used at trajectory timelines such that $t_j \le T_\ell < t_{j+1}$. For $\dot{\Delta t}$ the same interval applies and the $\dot{\Delta t}$ correction is converted to a correction to Δt by multiplying it by $T_\ell - t_0$.

Improved initial conditions are computed if required after trajectory propagation is completed. For each mode of operation the computations are slightly different. In all modes the improved inertial position and velocity are obtained by interpolating off of the propagated trajectory at the selected times. In the "batch" mode the ΔRP_N values are added to the nominal radiation pressure parameter values to get the improved radiation pressure parameter values–the same for all initial condition times. A similar computation is done for thrust parameters. In the "Filter" mode both the ΔK_{R_j} and ΔRP_j values just before the time for initial conditions are added to the nominal radiation pressure parameter values to get improved values. A similar computation is done for thrust parameters. In the "Smoother" mode the ΔRP_N values and the average ΔK_{R_j} values $\left(\dfrac{\sum_{j=0}^{N} \Delta K_{R_j}}{N+1}\right)$ are both added to the nominal radiation pressure parameter values to get improved values. These radiation pressure parameter values would normally be used to predict a reference trajectory for a future span. This is why the average stochastic radiation pressure corrections are used. The thrust corrections are added to the nominal thrust values to get improved values. Also the gravity coefficient parameter corrections are added to their nominal values to get improved values.

SATRACK COVARIANCE MATRIX GENERATION

Special covariance matrices are required for the SATRACK application of the MSF/S system of programs. These matrices relate 8 parameters (position, velocity, time offset, and frequency offset) for each satellite to the same parameters for every satellite at a given time (intersatellite covariances) and at up to three different times (intertime covariances). The full covariance matrix required is structured as follows:

$$
P_{SATRACK} = \begin{array}{c} \begin{array}{ccc} \overbrace{}^{8N_{SV}} & \overbrace{}^{8N_{SV}} & \overbrace{}^{8N_{SV}} \end{array} \\ \left(\begin{array}{c:c:c} P_{11} & P_{12} & P_{13} \\ \hdashline P_{21} & P_{22} & P_{23} \\ \hdashline P_{31} & P_{32} & P_{33} \end{array} \right) \begin{array}{l} \}\, 8N_{SV} \\ \}\, 8N_{SV} \\ \}\, 8N_{SV} \end{array} \end{array} \tag{148}
$$

where the subscripts refer to times T_1, T_2, and T_3 and N_{SV} is the number of satellites. P_{11}, P_{22}, and P_{33} are symmetric submatices and $P_{21} = P_{12}^T$, $P_{31} = P_{13}^T$, and $P_{32} = P_{23}^T$. Each submatrix is further divided into 8×8 blocks where blocks along the diagonal relate a satellite to itself either at the same time or at two different times. Each 8×8 block above or below the diagonal blocks in each submatrix relate one satellite to another satellite either at the same time or at two different times. The diagonal submatrices P_{11}, P_{22}, and P_{33} are computed exactly but the off-diagonal submatrices P_{12}, P_{13}, and P_{23} are approximated in such a way that they would be exact if the process noise terms were zero.

As indicated in Figure 3 in the **TIMELINE DEFINITIONS** section, the times of interest T_1, T_2, and T_3 may not be exactly on mini-batch steps. Each T_m is associated with the mini-batch interval such that $T_m \varepsilon \left(t_j - \frac{\Delta t^{Red.}}{2}, t_j + \frac{\Delta t^{Red.}}{2} \right]$. Let $R^{-1}(T_m)$, $m = 1,2,3$ denote the $R_j^{*\,'}$ matrices for these mini-batch times. Each matrix is of dimension $N_{TOT} \times N_{TOT}$ and is upper triangular. For each time T_m a state transition matrix $\phi(T_m)$ is defined as given below. This matrix maps the parameter corrections at t_j into corrections in position, velocity, satellite time offset, and satellite frequency offset at T_m.

$$
\phi(T_m) = \left(K_R \;\vdots\; G \;\vdots\; \underset{\substack{\uparrow \\ \text{corresponds} \\ \text{to } C_R}}{0} \;\vdots\; \tau \;\vdots\; \underset{\substack{\uparrow \\ \text{corresponds} \\ \text{to } C_{MS}}}{0} \;\vdots\; \Phi_e \;\vdots\; \underset{\substack{\uparrow \\ \text{corresponds} \\ \text{to } S}}{0} \;\vdots\; RP \;\vdots\; T \;\vdots\; PM \;\vdots\; GC \right) \}\, 8N_{SV} \tag{149}
$$

Columns are present only for the parameters included in $R^{-1}(T_m)$.

$$
\underset{8N_{SV}\times 3N_{SV}}{K_R} \;=\;
\begin{array}{cc}
 & \begin{array}{cccccc} \overbrace{\qquad}^{3} & \overbrace{\qquad}^{3} & & & & \end{array} \\
\begin{array}{c} 6\{ \\ 2\{ \\ \\ \\ \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
K_{R_1} & 0 & 0 & & & \\
0 & 0 & 0 & & & \\
0 & K_{R_2} & 0 & & 0 & \\
0 & 0 & 0 & & & \\
0 & 0 & K_{R_3} & & & \\
0 & 0 & 0 & \ddots & & \\
 & & & & K_{R_{N_{SV}}} & \\
 & 0 & & & 0 &
\end{pmatrix}
\end{array}
\qquad (150)
$$

G and RP have the same size and structure as K_R

K_{R_i} = partials of position and velocity at T_m with respect to stochastic radiation pressure parameters at t_j given by equations (22) and (23) in the STATE EQUATIONS section with t_{j+1} replaced by T_m

G_i = partials of position and velocity at T_m with respect to gravitational acceleration parameters at t_j given by equations (29) and (30) in the STATE EQUATIONS section with t_{j+1} replaced by T_m

RP_i = partials of position and velocity at t_j with respect to radiation pressure parameters obtained by interpolating off of the trajectory at T_m

$$
\underset{8N_{SV}\times 3N_{SV}}{\tau} \;=\;
\begin{array}{cc}
 & \begin{array}{cccccc} \overbrace{\qquad}^{3} & \overbrace{\qquad}^{3} & & & & \end{array} \\
\begin{array}{c} 6\{ \\ 2\{ \\ \\ \\ \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
0 & 0 & 0 & & & \\
\tau_1 & 0 & 0 & & & \\
0 & 0 & 0 & & 0 & \\
0 & \tau_2 & 0 & & & \\
0 & 0 & 0 & & & \\
0 & 0 & \tau_3 & \ddots & & \\
 & & & & 0 & \\
 & 0 & & & \tau_{N_{SV}} &
\end{pmatrix}
\end{array}
\qquad (151)
$$

$$
\tau_i = \begin{pmatrix} \dfrac{T_m^2}{2} & T_m & 1 \\ T_m & 1 & 0 \end{pmatrix} \qquad i = 1,2,\ldots,N_{SV} \qquad (152)
$$

$$\underset{8N_{SV}\times 6N_{SV}}{\Phi_e} = \begin{array}{r l} & \begin{matrix} \overbrace{\quad}^{6} & \overbrace{\quad}^{6} & & & \end{matrix} \\ \begin{matrix} 6\{ \\ 2\{ \\ \\ \\ \\ \\ \\ \\ \end{matrix} & \begin{pmatrix} \Phi_{e_1} & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & \Phi_{e_2} & 0 & & 0 \\ 0 & 0 & 0 & & \\ 0 & 0 & \Phi_{e_3} & & \\ 0 & 0 & 0 & \ddots & \\ & & & & \Phi_{e_{N_{SV}}} \\ & 0 & & & 0 \end{pmatrix} \end{array} \tag{153}$$

Φ_{e_i} = partials of position and velocity at T_m with respect to orbital elements obtained by interpolating off of the trajectory at T_m

$$\underset{8N_{SV}\times 3N_{NT}}{T} = \begin{array}{r l} & \begin{matrix} \overbrace{\quad}^{3} & & \end{matrix} \\ \begin{matrix} 6\{ \\ 2\{ \\ \\ \\ \\ \\ \end{matrix} & \begin{pmatrix} T_1 & & \\ 0 & & 0 \\ & T_2 & \\ & 0 & \\ & & T_{N_T} \\ & 0 & 0 \end{pmatrix} \end{array} \tag{154}$$

The rows for each T_i are determined by which satellites have thrusts.

T_i = partials of position and velocity at T_m with respect to thrust computed using equation (79) in the Range Partial Derivatives subsection except evaluated at T_m and all 6 rows are required

$$\underset{8N_{SV}\times N_{PM}}{PM} = \begin{array}{cc} N_{PM} & \\ \begin{pmatrix} PM_1 \\ 0 \\ PM_2 \\ 0 \\ PM_3 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ PM_{N_{SV}} \\ 0 \end{pmatrix} & \begin{matrix} \}\,6 \\ \}\,2 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \end{array} \tag{155}$$

GC has the same size and structure as PM

PM_i = partials of position and velocity at T_m with respect to polar motion computed as in equation (80) in the Range Partial Derivatives subsection except evaluated at T_m and $\dot{r}$ replaces r in computing the velocity partials

GC_i = partials of position and velocity at T_m with respect to gravity coefficients obtained by interpolating off of the trajectory at T_m

The following product matrices are then formed:

$$\underset{8N_{SV}\times N_{TOT}}{S(T_m)} = \underset{8N_{SV}\times N_{TOT}}{\phi(T_m)}\,\underset{N_{TOT}\times N_{TOT}}{R^{-1}(T_m)} \qquad m=1,2,3 \tag{156}$$

Then the covariance submatrices of $P_{SATRACK}$ are given as follows:

$$P_{11} = S(T_1)S(T_1)^T \tag{157}$$

$$P_{12} = S(T_1)S(T_2)^T \tag{158}$$

$$P_{13} = S(T_1)S(T_3)^T \tag{159}$$

$$P_{22} = S(T_2)S(T_2)^T \tag{160}$$

$$P_{23} = S(T_2)S(T_3)^T \tag{161}$$

$$P_{33} = S(T_3)S(T_3)^T \tag{162}$$

Each is an $8N_{SV}\times 8N_{SV}$ matrix where the eight parameters are position, velocity, satellite time offset, and satellite frequency offset in units of km, km/sec, µsec, and ppm respectively. These matrices are then scaled to be in units of m, m/sec, sec, and sec/sec.

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APPENDIX A

STATE EQUATION SIMPLIFICATIONS

The purpose of this appendix is to describe the assumptions and definitions made in adopting the specialized form of the state equations for the orbit-related parameters in the MSF/S system. The first simplification involves assumptions about the white noise sources driving the estimated orbit-related states and affects the form of the process noise covariance matrix. The second simplification involves the definition of the pseudoepoch state variables and affects both the state transition matrices required for the Filter propagation step and the observational equation partial derivatives required for the Filter measurement update step.

To simplify this discussion, consider the set of state equations for a single satellite that includes current state position and velocity parameters, $\mathbf{x}'$, and one stochastic orbit-related parameter, p. The system of stochastic differential equations describing corrections to these parameters is given by

$$\dot{\begin{pmatrix} \delta p \\ \delta \mathbf{x}' \end{pmatrix}} = \begin{pmatrix} B & 0^T \\ F'_p & F'_x \end{pmatrix} \begin{pmatrix} \delta p \\ \delta \mathbf{x}' \end{pmatrix} + \begin{pmatrix} w_1^c \\ \mathbf{w}_2^c \end{pmatrix} \tag{A1}$$

where $B = -\dfrac{1}{\tau}$, i.e., δp is modeled as a first-order Gauss-Markov process (see Appendix B)

$F'_p = \dfrac{\partial \dot{\mathbf{x}}'}{\partial p}$ = partial derivatives of velocity and acceleration with respect to p

$F'_x = \dfrac{\partial \dot{\mathbf{x}}'}{\partial \mathbf{x}'}$ = partial derivatives of velocity and acceleration with respect to position and velocity

and it is assumed that changes in $\mathbf{x}'$ cause negligible changes in p. Also each $\mathbf{w}^c$ component is a white noise process. Assume that $\mathbf{w}_2^c = \mathbf{0}$, i.e., there is no white noise driving the $\delta\mathbf{x}'$ states directly. Then the discrete system equivalent to equation (A1) is given by

$$\begin{pmatrix} \Delta p \\ \Delta \mathbf{x}' \end{pmatrix}_{j+1} = \begin{pmatrix} M & 0^T \\ V'_{p_j} & V'_{x_j} \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta \mathbf{x}' \end{pmatrix}_j + \begin{pmatrix} w_1 \\ \mathbf{w}_2 \end{pmatrix}_j \tag{A2}$$

where $M = e^{-\frac{(t_{j+1}-t_j)}{\tau}}$ (see Appendix B)

$V'_{p_j} = V'_p(t_{j+1}, t_j) = \dfrac{\partial \mathbf{x}'(t_{j+1})}{\partial p(t_j)}$ = partial derivatives of position and velocity at t_{j+1} with respect to p at t_j

$V'_{x_j} = V'_x(t_{j+1}, t_j) = \dfrac{\partial \mathbf{x}'(t_{j+1})}{\partial \mathbf{x}'(t_j)}$ = partial derivatives of position and velocity at t_{j+1} with respect to position and velocity at t_j

$$\text{and} \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_j = \int_{t_j}^{t_{j+1}} \begin{pmatrix} e^{-\frac{(\lambda - t_j)}{\tau}} & 0^T \\ V_p'(\lambda, t_j) & V_x'(\lambda, t_j) \end{pmatrix} \begin{pmatrix} w_1^c \\ 0 \end{pmatrix} d\lambda$$

$$= \int_{t_j}^{t_{j+1}} \begin{pmatrix} e^{-\frac{(\lambda - t_j)}{\tau}} \, w_1^c \\ V_p'(\lambda, t_j) \, w_1^c \end{pmatrix} d\lambda \qquad \text{(A3)}$$

Therefore, $\boldsymbol{w}_2$ is a non-zero vector as a result of the discretization process even though the white noise driving the $\delta \boldsymbol{x}'$ states in the continuous system is assumed to be zero. The state equations adopted to simplify the square root information filter/smoother algorithms also ignore the $\boldsymbol{w}_2$ noise contributions. However, the $\Delta \boldsymbol{x}'$ states are still smoothable since they are dynamically related to the Δp state through the V_p' matrix. This assumption results in the absence of both a process noise covariance matrix for the six orbit states and a process noise cross-covariance matrix between these six states and the stochastic orbit-related states. Since the orbit states are non-stochastic (not p parameters), a considerable savings in array storage results. This is the case because the information array required for the Filter propagation step (equation (111)) includes two rows and columns for each p parameter. This reduction therefore allows more satellites to be processed simultaneously.

Expanding the state equations given in equation (A2) to include all orbit-related stochastic states, $\Delta \boldsymbol{p}$, and orbit-related bias states, $\Delta \mathbf{y}$, and again assuming $\boldsymbol{w}_2 = \boldsymbol{0}$, gives the following state equations:

$$\begin{pmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{x}' \\ \Delta \mathbf{y} \end{pmatrix}_{j+1} = \begin{pmatrix} M & 0 & 0 \\ V_{p_j}' & V_{x_j}' & V_{y_j}' \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{x}' \\ \Delta \mathbf{y} \end{pmatrix}_j + \begin{pmatrix} \boldsymbol{w}_j \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \qquad \text{(A4)}$$

where $M = \text{diag}\left(e^{-\frac{(t_{j+1} - t_j)}{\tau_i}} \right)$

$V_{p_j}' = V_p'(t_{j+1}, t_j)$ = partial derivatives of position and velocity at t_{j+1} with respect to $\boldsymbol{p}$ at t_j

$V_{x_j}' = V_x'(t_{j+1}, t_j)$ = partial derivatives of position and velocity at t_{j+1} with respect to position and velocity at t_j

and $V_{y_j}' = V_y'(t_{j+1}, t_j)$ = partial derivatives of position and velocity at t_{j+1} with respect to $\boldsymbol{y}$ at t_j.

The subset of these equations involving the $\Delta x'$ states is given by

$$\Delta \mathbf{x}'(t_{j+1}) = V'_p\,(t_{j+1}, t_j)\,\Delta \mathbf{p}(t_j) + V'_x(t_{j+1}, t_j)\,\Delta \mathbf{x}'(t_j) + V'_y(t_{j+1}, t_j)\,\Delta \mathbf{y}(t_j) \tag{A5}$$

Define the pseudoepoch state variables $\Delta \mathbf{x}_j$ by

$$\Delta \mathbf{x}'(t_j) = V_x(t_j, T_0)\,\Delta \mathbf{x}_j + V_y(t_j, T_0)\,\Delta \mathbf{y}(t_j) \tag{A6}$$

where $V_x(t_j,T_0)$ = partial derivatives of position and velocity at t_j with respect to either orbital elements or position and velocity at the trajectory epoch T_0

and $V_y(t_j,T_0)$ = partial derivatives of position and velocity at t_j with respect to the orbit-related bias parameters at T_0.

Substituting for $\Delta \mathbf{x}'(t_j)$ and $\Delta \mathbf{x}'(t_{j+1})$ from equation (A6) into equation (A5) results in the following equation:

$$V_x(t_{j+1},T_0)\,\Delta \mathbf{x}_{j+1} + V_y\,(t_{j+1}, T_0)\,\Delta \mathbf{y}(t_{j+1}) = \tag{A7}$$
$$V'_p(t_{j+1},t_j)\,\Delta \mathbf{p}(t_j) + V'_x\,(t_{j+1}, t_j)V_x(t_j,T_0)\Delta \mathbf{x}_j + V'_x(t_{j+1}, t_j)V_y(t_j,T_0)\Delta \mathbf{y}(t_j) + V'_y\,(t_{j+1}, t_j)\Delta \mathbf{y}(t_j)$$

Multiplying both sides of equation (A7) by $V_x^{-1}(t_{j+1}, T_0)$ and rearranging terms results in the following equation:

$$\begin{aligned}\Delta \mathbf{x}_{j+1} = {} & V_x^{-1}(t_{j+1},T_0)\,V'_x(t_{j+1},t_j)V_x(t_j,T_0)\Delta \mathbf{x}_j + V_x^{-1}(t_{j+1}, T_0)V'_p\,(t_{j+1},t_j)\Delta \mathbf{p}(t_j) \\ & + V_x^{-1}(t_{j+1},T_0)\Big[\big(V'_x(t_{j+1},t_j)V_y(t_j,T_0) + V'_y(t_{j+1},t_j)\big)\Delta \mathbf{y}(t_j) - V_y\,(t_{j+1},T_0)\Delta \mathbf{y}(t_{j+1})\Big]\end{aligned} \tag{A8}$$

Based on the properties of state transition matrices, the following identity exists:

$$V'_x(t_{j+1},t_j)V_x(t_j,T_0) = V_x\,(t_{j+1},T_0) \tag{A9}$$

Therefore the matrix multiplying $\Delta \mathbf{x}_j$ in equation (A8) reduces to an identity matrix. Also, based on the properties of state transition matrices and $\Delta \mathbf{y}(t_{j+1}) = \Delta \mathbf{y}(t_j)$ from equation (A4), the following identity exists:

$$[V'_x(t_{j+1},t_j)V_y(t_j,T_0) + V'_y(t_{j+1},t_j)]\,\Delta \mathbf{y}(t_j) = V_y\,(t_{j+1},T_0)\Delta \mathbf{y}(t_{j+1}) \tag{A10}$$

Therefore the last term in equation (A8) vanishes and equation (A8) reduces to:

$$\begin{aligned}\Delta \mathbf{x}_{j+1} &= \Delta \mathbf{x}_j + V_x^{-1}(t_{j+1},T_0)V'_p\,(t_{j+1},t_j)\Delta \mathbf{p}\,(t_j) \\ &= \Delta \mathbf{x}_j + V_{p_j}\Delta \mathbf{p}\,(t_j)\end{aligned} \tag{A11}$$

where $V_{p_j} = V_x^{-1}(t_{j+1},T_0)V'_p\,(t_{j+1},t_j)$ (A12)

Therefore the state equations given by equation (A4) and the pseudoepoch state variables definition given by equation (A6) result in the following simplified state equations for orbit-related parameters:

$$\begin{pmatrix} \Delta p \\ \Delta x \\ \Delta y \end{pmatrix}_{j+1} = \begin{pmatrix} M & 0 & 0 \\ V_{p_j} & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta x \\ \Delta y \end{pmatrix}_{j} + \begin{pmatrix} w_j \\ 0 \\ 0 \end{pmatrix} \qquad \text{(A13)}$$

To be consistent with these equations, the observational equation partial derivatives required in the Filter measurement update step must be computed with respect to these variables. For a given observation at time t_{obs}, the partials with respect to the orbit-related p parameters involve the $V_p^j(t_{obs}, t_j)$ matrix, the partials with respect to the pseudoepoch state orbit parameters involve the $V_x(t_{obs}, T_0)$ matrix, and the partials with respect to the orbit-related bias parameters involve the $V_y(t_{obs}, T_0)$ matrix.

APPENDIX B

FIRST-ORDER GAUSS-MARKOV PROCESSES

The stochastic differential equation defining a continuous first-order Gauss-Markov process is given by

$$\dot{\delta p} = -\frac{1}{\tau}\delta p + w \tag{B1}$$

where τ = decorrelation time (seconds)

w = white Gaussian noise of mean zero and spectral density q, i.e., $E(w) = 0$ and $E[w(t)w(s)] = q\delta(t-s)$.

The continuous linear variance differential equation corresponding to this stochastic process is given by

$$\dot{\sigma}^2 = -\frac{2}{\tau}\sigma^2 + q \tag{B2}$$

Setting the variance rate, $\dot{\sigma}^2$, to zero and solving for σ^2 gives the steady-state variance for this random process as

$$\sigma^2 = E(\delta p(t)^2) = \frac{\tau}{2}q \tag{B3}$$

Also the mean value, μ, of this process is zero, i.e.,

$$\mu = E(\delta p(t)) = 0 \tag{B4}$$

The one-sided power spectral density for a Gauss-Markov process is given by

$$\Phi(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2} \tag{B5}$$

where $\beta = \dfrac{1}{\tau}$ and $\omega = 2\pi f$.

Its corresponding autocorrelation function is given by

$$\phi(t) = \sigma^2 e^{-\beta|t|} \tag{B6}$$

A Gauss-Markov process is used to approximate a band-limited process with a flat spectral density over this bandwidth.

The discrete equivalent of this continuous process is given by the first-order stochastic difference equation

$$\Delta p_{j+1} = \phi\Delta p_j + w_j \tag{B7}$$

where ϕ = state transition matrix

$\{w_j\}$ = white Gaussian noise sequence of mean zero and variance Q, i.e., $E(w_j) = 0$ and $E(w_j w_k) = Q\delta(j-k)$.

ϕ satisfies the following differential equation and initial condition:

$$\dot{\phi} = -\frac{1}{\tau}\phi \tag{B8}$$

$$\phi(t_j, t_j) = 1 \tag{B9}$$

The solution to this differential equation is simply

$$\phi = e^{-\frac{(t_{j+1}-t_j)}{\tau}} = e^{-\frac{\Delta t}{\tau}} \tag{B10}$$

The process noise variance, Q, is then given by

$$\begin{aligned} Q &= \int_{t_j}^{t_{j+1}} \phi(\lambda, t_j) q \phi^T(\lambda, t_j) d\lambda = q \int_{t_j}^{t_{j+1}} e^{-\frac{2(\lambda - t_j)}{\tau}} d\lambda \\ &= -q\frac{\tau}{2} e^{-\frac{2(\lambda-t_j)}{\tau}} \Bigg|_{t_j}^{t_{j+1}} = q\frac{\tau}{2}\left(1 - e^{-\frac{2\Delta t}{\tau}}\right) \\ &= \sigma^2 \left(1 - e^{-\frac{2\Delta t}{\tau}}\right) \end{aligned} \tag{B11}$$

For $\tau = \infty$, the stochastic differential equation (B1) reduces to that for a random walk process, i.e.,

$$\dot{\delta p} = w \tag{B12}$$

The discrete equivalent is then given by

$$\Delta p_{j+1} = \Delta p_j + w_j \tag{B13}$$

For this case Q is derived starting with equation (B11) as follows:

$$\begin{aligned} Q &= \lim_{\tau \to \infty} q \frac{\tau}{2} \left(1 - e^{-\frac{2\Delta t}{\tau}}\right) \\ &= q \lim_{\tau \to \infty} \frac{\tau}{2} \left[1 - \left(1 - \frac{2\Delta t}{\tau} + \frac{\left(\frac{2\Delta t}{\tau}\right)^2}{2!} - \ldots\right)\right] \end{aligned} \tag{B14}$$

$$= q \lim_{\tau \to \infty} \left(\Delta t + \sum_{i=1}^{\infty} constant_i \left(\frac{1}{\tau}\right)^i \right)$$

$$= q\Delta t$$

If, in addition, $q=0$, then the stochastic differential equation (B12) reduces to that for a random constant, i.e.,

$$\dot{\delta p} = 0 \tag{B15}$$

The discrete equivalent is then given by

$$\Delta p_{j+1} = \Delta p_j \tag{B16}$$

and $$Q = 0 \tag{B17}$$

APPENDIX C

APPROXIMATION OF PARTIAL DERIVATIVES OF POSITION AND VELOCITY WITH RESPECT TO STOCHASTIC ORBIT-RELATED STATES

The partial derivatives of position and velocity at t with respect to the stochastic radiation pressure and gravitational acceleration parameters at t_j are approximated using a Taylor series expansion method. These partials only have to be accurate for values of $t \le t_{j+1}$, i.e., for a time span of at most one mini-batch interval (usually ≤ 1 hour). Using p to designate the stochastic parameters (either K_R or G) and r and $\dot{r}$ to designate position and velocity respectively, the system of differential equations defining the corrections to the parameters (with the stochastic terms set to zero) used to derive the state transition matrix containing the required partial derivatives is given by

$$\begin{pmatrix} \delta \dot{p} \\ \delta \dot{r} \\ \delta \ddot{r} \end{pmatrix} = \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & I \\ F_1 & G & 0 \end{pmatrix} \begin{pmatrix} \delta p \\ \delta r \\ \delta \dot{r} \end{pmatrix} \tag{C1}$$

where $B = \text{diag}\left(-\frac{1}{\tau_i}\right)$ i.e., each component of δp is modeled as an independent Gauss-Markov process (see Appendix B)

$F_1 = \frac{\partial \ddot{r}}{\partial p}$ = partial derivatives of acceleration with respect to p

and $G = \frac{\partial \ddot{r}}{\partial r}$ = partial derivatives of acceleration with respect to r

It is assumed that changes in r and $\dot{r}$ cause negligible changes in p and that acceleration is not a function of velocity.
Define

$$F = \begin{pmatrix} B & 0 \\ F_2 & F_3 \end{pmatrix} \tag{C2}$$

where $$F_2 = \begin{pmatrix} 0 \\ F_1 \end{pmatrix} \tag{C3}$$

and $$F_3 = \begin{pmatrix} 0 & I \\ G & 0 \end{pmatrix} \tag{C4}$$

The state transition matrix, $\phi(t,t_j)$, is the solution of the following system of differential equations and initial conditions:

$$\dot{\phi}(t,t_j) = F(t)\phi(t,t_j) \tag{C5}$$

$$\phi(t_j,t_j) = I \tag{C6}$$

Approximating $\phi(t,t_j)$ by a second-order Taylor series about t_j and assuming $\dot{F}=0$, i.e., F is constant for short time intervals, results in the following:

$$\phi(t,t_j) \doteq \phi(t_j,t_j)+\dot{\phi}(t_j,t_j)(t-t_j)+\ddot{\phi}(t_j,t_j)\frac{(t-t_j)^2}{2} \tag{C7}$$

$$= I+F(t_j)\phi(t_j,t_j)(t-t_j)+\left[\dot{F}(t_j)\phi(t_j,t_j)+F(t_j)F(t_j)\phi(t_j,t_j)\right]\frac{(t-t_j)^2}{2}$$

$$= I+F(t_j)(t-t_j)+F^2(t_j)\frac{(t-t_j)^2}{2}$$

where
$$F^2(t_j) = \begin{pmatrix} B^2 & 0 \\ F_3(t_j)F_2(t_j)+F_2(t_j)B & F_2^2(t_j) \end{pmatrix} \tag{C8}$$

Partitioning ϕ the same way that F was partitioned in equation (C2) gives

$$\phi(t,t_j) = \begin{pmatrix} \phi_1(t,t_j) & 0 \\ \phi_2(t,t_j) & \phi_3(t,t_j) \end{pmatrix} \tag{C9}$$

Then
$$\phi_1(t,t_j) \doteq I+B(t-t_j)+B^2\frac{(t-t_j)^2}{2} \tag{C10}$$

However, in the limit as the number of terms increases, the right-hand side of this expression converges to $\mathrm{diag}\left(e^{-\frac{(t-t_j)}{\tau_i}}\right)$ so no approximation is necessary. Also,

$$\phi_3(t,t_j) \doteq I+F_3(t_j)(t-t_j)+F_2^2(t_j)\frac{(t-t_j)^2}{2} \tag{C11}$$

This approximation is also not required since $\phi_3(t, t_j)$ is just the partials of position and velocity at t with respect to position and velocity at t_j, which can be obtained exactly by proper manipulation of partials interpolated off of the trajectory. Therefore, the only submatrix of $\phi(t,t_j)$ requiring approximation is $\phi_2(t,t_j)$ and it is given by

$$\phi_2(t,t_j) \doteq F_2(t_j)(t-t_j)+\left(F_3(t_j)F_2(t_j)+F_2(t_j)B\right)\frac{(t-t_j)^2}{2} \tag{C12}$$

$$= \begin{pmatrix} 0 \\ F_1(t_j) \end{pmatrix}(t-t_j)+\left[\begin{pmatrix} F_1(t_j) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ F_1(t_j)B \end{pmatrix}\right]\frac{(t-t_j)^2}{2}$$

Therefore

$$\frac{\partial r(t_j)}{\partial p(t_j)} \doteq F_1(t_j)\,\frac{(t-t_j)^2}{2} \tag{C13}$$

and
$$\frac{\partial \dot{r}(t_j)}{\partial p(t_j)} \doteq F_1(t_j)(t-t_j) + F_1(t_j)B\,\frac{(t-t_j)^2}{2} \tag{C14}$$

For the radiation pressure (K_R) parameters, $F_1(t_j) = \dfrac{\partial \ddot{r}(t_j)}{\partial K_R(t_j)}$ is computed as follows:

$$\ddot{r}(t_j) = R_s \begin{pmatrix} K_{R_1} a_x^M + K_{R_2} 10^{-12}\,shape\cos K_{R_3} \\ K_{R_2} 10^{-12}\,shape\sin K_{R_3} \\ K_{R_1} a_z^M \end{pmatrix} \tag{C15}$$

$$\frac{\partial \ddot{r}(t_j)}{\partial K_R(t_j)} = R_s \begin{pmatrix} a_x^M & 10^{-12}\,shape\cos K_{R_3} & -K_{R_2}10^{-12}\,shape\sin K_{R_3} \\ 0 & 10^{-12}\,shape\sin K_{R_3} & K_{R_2}10^{-12}\,shape\cos K_{R_3} \\ a_z^M & 0 & 0 \end{pmatrix} \tag{C16}$$

where R_s = matrix required to transform between the body-axis and inertial Cartesian reference systems obtained from the trajectory at t_j

$$a_x^M = \frac{a_x - K_{R_2}10^{-12}\,shape\cos K_{R_3}}{K_{R_1}}, \quad a_z^M = \frac{a_z}{K_{R_1}} \tag{C17}$$

= inertial accelerations due to the radiation pressure model only and not including y-axis and K_{R_1} contributions, given in the body-axis x and z directions respectively at t_j.

$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ = inertial acceleration at t_j due to radiation pressure in the body-axis directions obtained from the trajectory

K_R = nominal radiation pressure parameter values from the trajectory

shape = fraction of the sun's disk unobstructed by any eclipsing body (Earth, Moon, or both) obtained from the trajectory at t_j

For the gravitational acceleration (G) parameters, $F_1(t_j) = \dfrac{\partial \ddot{r}(t_j)}{\partial G(t_j)}$ is computed as follows:

$$\ddot{r}(t_j) = R_{RAC}G \tag{C18}$$

$$\frac{\partial \ddot{\mathbf{r}}(t_j)}{\partial \mathbf{G}(t_j)} = R_{RAC} \tag{C19}$$

where $$R_{RAC} = \left(\hat{\mathbf{r}} \quad \frac{\hat{\mathbf{r}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{r}} \times \hat{\mathbf{v}}|} \times \hat{\mathbf{r}} \quad \frac{\hat{\mathbf{r}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{r}} \times \hat{\mathbf{v}}|} \right) \tag{C20}$$

= matrix required to transform between the *RAC* and inertial Cartesian reference frames at t_j

$\mathbf{r},\mathbf{v}$ = position and velocity at t_j interpolated off of the trajectory (^ denotes a unit vector)

APPENDIX D
CLOCK MODELS

The current state satellite clock offsets are modeled as a third-order linear system with white noise inputs given by the following stochastic differential equations:

$$\frac{d}{dt}\begin{pmatrix} \delta\ddot{\tau} \\ \delta\dot{\tau} \\ \delta\tau \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \delta\ddot{\tau} \\ \delta\dot{\tau} \\ \delta\tau \end{pmatrix} + \begin{pmatrix} w_1'' \\ w_2'' \\ w_3'' \end{pmatrix} \tag{D1}$$

where $\delta\ddot{\tau}$ = frequency drift

$\delta\dot{\tau}$ = frequency offset

$\delta\tau$ = time offset

w_i'' = white noise process of mean zero and spectral density q_i, i.e., $E(w_i''(t))=0$ and $E(w_i''(t)w_i''(s))=q_i\,\delta(t-s)$, $i=1,2,3$

The discrete equivalent of this continuous model is given by

$$\begin{pmatrix} \Delta\ddot{\tau} \\ \Delta\dot{\tau} \\ \Delta\tau \end{pmatrix}_{j+1} = \begin{pmatrix} 1 & 0 & 0 \\ \Delta t & 1 & 0 \\ \frac{\Delta t^2}{2} & \Delta t & 1 \end{pmatrix} \begin{pmatrix} \Delta\ddot{\tau} \\ \Delta\dot{\tau} \\ \Delta\tau \end{pmatrix}_{j} + \begin{pmatrix} w_1' \\ w_2' \\ w_3' \end{pmatrix}_{j} \tag{D2}$$

where $\Delta t = t_{j+1} - t_j$ (D3)

$\{w_j'\}$ = white noise vector sequence of mean zero and covariance matrix Q derived as follows:

$$Q = \int_0^{\Delta t} \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \frac{\lambda^2}{2} & \lambda & 1 \end{pmatrix} \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_2 \end{pmatrix} \begin{pmatrix} 1 & \lambda & \frac{\lambda^2}{2} \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} d\lambda \tag{D4}$$

$$= \int_0^{\Delta t} \begin{pmatrix} q_1 & q_1\lambda & q_1\frac{\lambda^2}{2} \\ q_1\lambda & q_1\lambda^2+q_2 & q_1\frac{\lambda^3}{2}+q_2\lambda \\ q_1\frac{\lambda^2}{2} & q_1\frac{\lambda^3}{2}+q_2\lambda & q_1\frac{\lambda^4}{4}+q_2\lambda^2+q_3 \end{pmatrix} d\lambda$$

$$= \begin{pmatrix} q_1\Delta t & q_1\frac{\Delta t^2}{2} & q_1\frac{\Delta t^3}{6} \\ q_1\frac{\Delta t^2}{2} & q_1\frac{\Delta t^3}{3} + q_2\Delta t & q_1\frac{\Delta t^4}{8} + q_2\frac{\Delta t^2}{2} \\ q_1\frac{\Delta t^3}{6} & q_1\frac{\Delta t^4}{8} + q_2\frac{\Delta t^2}{2} & q_1\frac{\Delta t^5}{20} + q_2\frac{\Delta t^3}{3} + q_3\Delta t \end{pmatrix}$$

In terms of the pseudoepoch state clock offsets actually implemented in the MSF/S system, the model is given by

$$\begin{pmatrix} \Delta\ddot{\tau}_0 \\ \Delta\dot{\tau}_0 \\ \Delta\tau_0 \end{pmatrix}_{j+1} = \begin{pmatrix} \Delta\ddot{\tau}_0 \\ \Delta\dot{\tau}_0 \\ \Delta\tau_0 \end{pmatrix}_j + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}_j \qquad \text{(D5)}$$

where the subscript o denotes a correspondence with the fit span epoch, t_0, and

$\{\boldsymbol{w}_j\}$ = white noise vector sequence of mean zero and covariance matrix Q_{0_j} derived from Q as follows:

Let $$\Delta\tau_0 = \begin{pmatrix} \Delta\ddot{\tau}_0 \\ \Delta\dot{\tau}_0 \\ \Delta\tau_0 \end{pmatrix},\ \Delta\tau = \begin{pmatrix} \Delta\ddot{\tau} \\ \Delta\dot{\tau} \\ \Delta\tau \end{pmatrix},\ \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},\ \text{and } \Phi_j = \begin{pmatrix} 1 & 0 & 0 \\ t_j - t_0 & 1 & 0 \\ \frac{(t_j-t_0)^2}{2} & t_j - t_0 & 1 \end{pmatrix}$$

Then $$\Delta\tau_{j+1} = \Phi_{j+1}\Delta\tau_{0_{j+1}} = \Phi_{j+1}\Delta\tau_{0_j} + \Phi_{j+1}\boldsymbol{w}_j \qquad \text{(D6)}$$

The state transition matrix in equation (D2) is equivalent to $\Phi_{j+1}\Phi_j^{-1}$ so that equation (D2) can be rewritten as

$$\Delta\tau_{j+1} = \Phi_{j+1}\Phi_j^{-1}\Delta\tau_j + \boldsymbol{w}_j' \qquad \text{(D7)}$$

Combining equations (D6) and (D7) results in the following equality:

$$\Phi_{j+1}\Delta\tau_{0_j} + \Phi_{j+1}\boldsymbol{w}_j = \Phi_{j+1}\Phi_j^{-1}\Delta\tau_j + \boldsymbol{w}_j' \qquad \text{(D8)}$$

Multiplying each side by its transpose gives

$$\begin{aligned} &\Phi_{j+1}\Delta\tau_{0_j}\Delta\tau_{0_j}^T\Phi_{j+1}^T + \Phi_{j+1}\boldsymbol{w}_j\boldsymbol{w}_j^T\Phi_{j+1}^T + \Phi_{j+1}\Delta\tau_{0_j}\boldsymbol{w}_j^T\Phi_{j+1}^T + \Phi_{j+1}\boldsymbol{w}_j\Delta\tau_{0_j}^T\Phi_{j+1}^T = \\ &\quad \Phi_{j+1}\Phi_j^{-1}\Delta\tau_j\Delta\tau_j^T\Phi_j^{-T}\Phi_{j+1}^T + \boldsymbol{w}_j'\boldsymbol{w}_j'^T + \Phi_{j+1}\Phi_j^{-1}\Delta\tau_j\boldsymbol{w}_j'^T + \boldsymbol{w}_j'\Delta\tau_j^T\Phi_j^{-T}\Phi_{j+1}^T \end{aligned} \qquad \text{(D9)}$$

Taking expected values and replacing

$$E(\Delta\tau_{0_j}\ \Delta\tau_{0_j}^T)\ \text{by}\ P_{0_j},$$

$$E(w_j\, w_j^T)\ \text{by}\ Q_{0_j},$$

$$E(\Delta\tau_{0_j} w_j^T)\ ,\ E(w_j \Delta\tau_{0_j}^T)\ ,\ E(\Delta\tau_j\ w_j'^T),\ \text{and}\ E(w_j'\ \Delta\tau_j^T)\ \text{by}\ 0,$$

$$E(\Delta\tau_j\ \Delta\tau_j^T)\ \text{by}\ P_j,$$

and $\quad E(w_j' w_j'^T)$ by Q gives

$$\Phi_{j+1} P_{0_j} \Phi_{j+1}^T + \Phi_{j+1} Q_{0_j}\ \Phi_{j+1}^T = \Phi_{j+1}\ \Phi_j^{-1} P_j \Phi_j^{-T}\ \Phi_{j+1}^T + Q \tag{D10}$$

Multiplying each side of this equation on the left by Φ_{j+1}^{-1} and on the right by Φ_{j+1}^{-T} and substituting P_{0_j} for $\Phi_j^{-1} P_j \Phi_j^{-T}$ gives

$$P_{0_j} + Q_{0_j}\ =\ P_{0_j} +\ \Phi_{j+1}^{-1}\, Q\, \Phi_{j+1}^{-T} \tag{D11}$$

Therefore $$Q_{0_j}\ =\ \Phi_{j+1}^{-1}\, Q\, \Phi_{j+1}^{-T} \tag{D12}$$

The square root of $Q_{0_j}^{-1}$ is actually required by the Filter algorithm. The upper triangular square root, R, of Q^{-1} such that

$$Q^{-1} = R^T R\ \ \text{or}\ \ Q = R^{-1} R^{-T} \tag{D13}$$

is first computed using Cholesky decomposition (see Appendix E). Then combining equations (D12) and (D13) gives

$$Q_{0_j} =\ \Phi_{j+1}^{-1} R^{-1}\ R^{-T}\ \Phi_{j+1}^{-T} \tag{D14}$$

Inverting gives

$$Q_{0_j}^{-1} =\ \Phi_{j+1}^T R^T\, R\, \Phi_{j+1} \tag{D15}$$

Therefore the square root of $Q_{0_j}^{-1}$ is just $R\,\Phi_{j+1}$ where R is invariant (a function of Δt only) and Φ_{j+1} is lower triangular. The product $R\Phi_{j+1}$ is a full matrix and must be upper triangularized using Householder transformations (see Appendix G) before being used in the propagation array.

To be consistent with the clock model states as given in equation (D5), the observational partial derivatives required in the Filter measurement update step must be computed with respect to these pseudoepoch state parameters. For a given observation at time t_{obs}, the partials must therefore involve the third row of the Φ_j matrix with t_j replaced by t_{obs}.

The current state station clock offsets are modeled as a second-order linear system with white noise inputs, i.e., the same as the satellite clock model except that all $\delta\ddot{\tau}$-related terms are absent. For the discrete version of this model the Q matrix is given by

$$Q = \begin{pmatrix} q_1 \Delta t & q_1 \dfrac{\Delta t^2}{2} \\ q_1 \dfrac{\Delta t^2}{2} & q_1 \dfrac{\Delta t^3}{3} + q_2 \Delta t \end{pmatrix} \tag{D16}$$

where $q_1 =$ white noise spectral density for the continuous frequency offset state, $\delta\dot{\tau}$

and $q_2 =$ white noise spectral density for the continuous time offset state, $\delta\tau$.

The ϕ_j matrix is given by

$$\phi_j = \begin{pmatrix} 1 & 0 \\ t_j - t_0 & 1 \end{pmatrix} \tag{D17}$$

With these definitions, all of the discussion of the satellite clock model above applies to the station clock model also.

Assuming that $q_1 = 0$ for the satellite clock model, i.e., frequency drift is modeled as a random constant over the entire fit span, the model for the frequency offset state, $\Delta\dot{\tau}$, is equivalent to a constant plus the integral of the frequency drift state, $\Delta\ddot{\tau}$, plus the integral of white noise. The model for the time offset state, $\Delta\tau$, is then equivalent to a constant plus the integral of the frequency offset state, $\Delta\dot{\tau}$, plus the integral of white noise. Therefore, the noise terms integrated in determining the time offset state are integrated white noise (random walk frequency noise) and white frequency noise.

The spectral densities of these noise terms can be directly related to the Allan variance which is used to characterize the statistical frequency fluctuations of atomic and crystal clocks. The Allan variance, $\sigma_y^2(\tau)$, is defined by

$$\sigma_y^2(\tau) = \frac{1}{2} E[(\bar{y}_{k+1}(\tau) - \bar{y}_k(\tau))^2] \tag{D18}$$

$$\text{where } \bar{y}_k(\tau) = \frac{1}{\tau} \int_{t_k}^{t_k+\tau} y(t)dt = \text{average fractional frequency over } \tau \text{ seconds} \tag{D19}$$

$$\text{and } \quad y(t) = \frac{\dot{\phi}(t)}{2\pi \nu_0} = \text{instantaneous fractional frequency} \tag{D20}$$

= rate of change of phase ϕ divided by 2π times the nominal frequency ν_0.

The Allan variance corresponding to the model with $q_1 = 0$ is given by the sum of the individual Allan variances for each noise source as follows:

$$\sigma_y^2(\tau) = \frac{q_3}{\tau} + \frac{q_2 \tau}{3} \tag{D21}$$

The first term corresponds to the white frequency noise and the second term corresponds to the random walk frequency noise. This is depicted in Figure D1, which is a typical plot of the square root of the Allan variance. Flicker noise, which corresponds to a horizontal line on this plot, cannot be represented exactly by this model. However, the white noise term (with spectral density q_3) can be chosen to exactly match the left-hand portion of a theoretical Allan variance curve. Then the q_2 spectral density can be selected optimally so that the minimum point of the combined curve lies on the flicker noise portion of the theoretical curve.

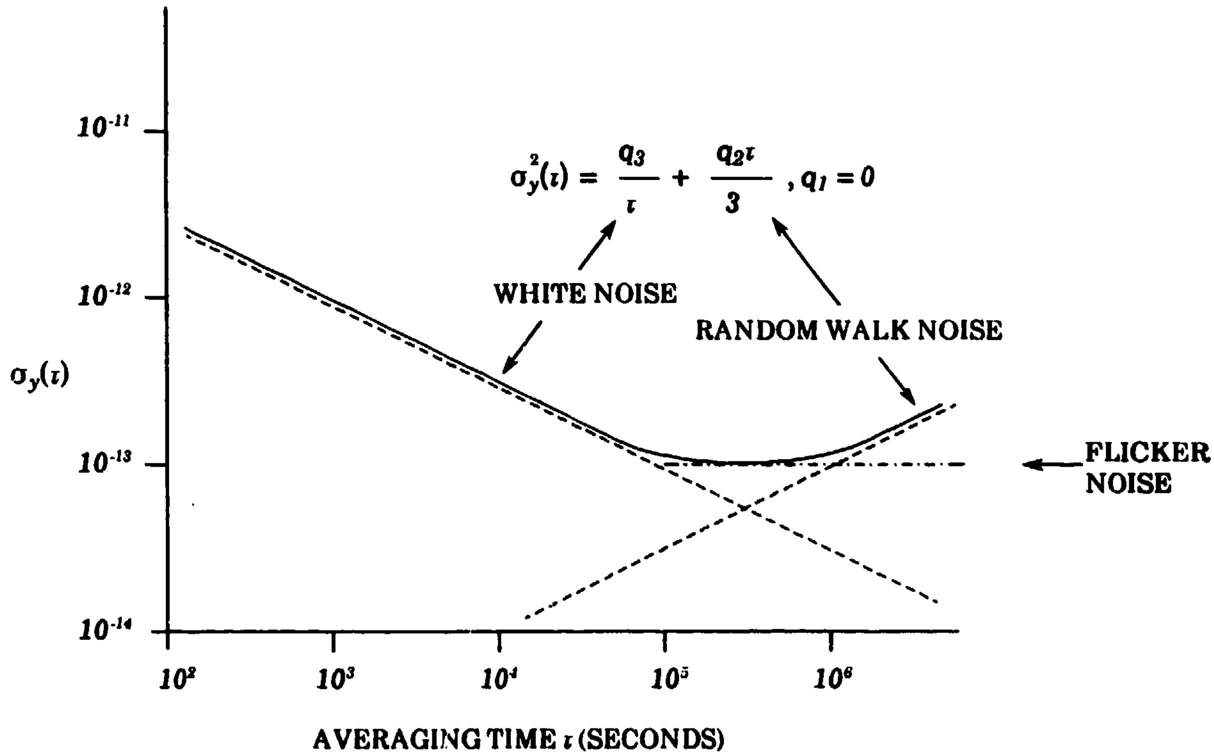

FIGURE D1. CORRESPONDENCE BETWEEN CLOCK MODEL SPECTRAL DENSITIES AND ALLAN VARIANCE

APPENDIX E

CHOLESKY DECOMPOSITION

Lower triangular Cholesky decomposition is used to compute the square root of the inverse of the process noise covariance matrix, Q, for each clock model, i.e., Q^{-1} is factored into the form $R^T R$ where R is upper triangular. The algorithm actually computes a lower triangular matrix $L = R^T$, i.e., $Q^{-1} = LL^T$. Let $q_{i,j} \varepsilon Q^{-1}(n \times n)$ and $\ell_{i,j} \varepsilon L(n \times n)$, $n = 2$ or 3. Then L is computed as follows starting with $j = 1$:

For column j define

$$\ell_{j,j} = (q_{j,j})^{\frac{1}{2}} \tag{E1}$$

If $j = n$, the procedure is complete.

Otherwise, for each subsequent row k define

$$\ell_{k,j} = q_{k,j}/\ell_{j,j} \qquad k = j+1 \ldots, n \tag{E2}$$

This completes the definition of this column of L and the Q^{-1} matrix is then adjusted as follows:

For all subsequent columns and all subsequent rows define

$$q_{i,k} = q_{i,k} - \ell_{i,j}\ell_{k,j} \qquad k = j+1 \ldots, n \qquad i = k \ldots, n \tag{E3}$$

Then go to the next column, $j+1$, and repeat this procedure.

APPENDIX F

WHITENING AND DECORRELATION OF RANGE DIFFERENCE OBSERVATIONS

Range difference observations are derived by one of two measurement techniques, both of which result in pairwise correlated observations. The first technique involves differencing two consecutive accumulated Doppler counts provided that the count was continuous, i.e., no losses of lock or cycle slips occurred during this interval. The second technique involves differencing two consecutive pseudorange measurements. In either case, two consecutive range difference observations (assuming no losses of lock or cycle slips for the Doppler-derived type) have one measurement in common and are therefore correlated.

Let RD represent a measurement-noise-free range difference observation and let ADR represent a noise-free Doppler count type measurement or pseudorange measurement. Then from three consecutive ADR measurements, two RD observations can be derived as follows:

$$RD_1 + \nu'_1 = (ADR_1 + \nu''_1) - (ADR_0 + \nu''_0) \tag{F1}$$

$$RD_2 + \nu'_2 = (ADR_2 + \nu''_2) - (ADR_1 + \nu''_1) \tag{F2}$$

where ν''_i represents the zero mean white measurement noise of variance $\sigma^2_{ADR_i}$ on the ADR measurements and ν'_i represents the resulting zero mean measurement noise on the RD observations. The RD observation measurement noise variances are derived by squaring each side of equations (F1) and (F2) and taking expected values to get the following:

$$\sigma^2_{RD_1} = E(\nu'^2_1) = E(\nu''^2_1) + E(\nu''^2_0) = \sigma^2_{ADR_1} + \sigma^2_{ADR_0} \tag{F3}$$

$$\sigma^2_{RD_2} = E(\nu'^2_2) = E(\nu''^2_2) + E(\nu''^2_1) = \sigma^2_{ADR_2} + \sigma^2_{ADR_1} \tag{F4}$$

The measurement noise covariance is derived by multiplying equation (F2) by equation (F1) and taking expected values to get the following:

$$\sigma_{RD_1,RD_2} = E(\nu'_1\nu'_2) = -E(\nu''^2_1) = -\sigma^2_{ADR_1} \tag{F5}$$

Then the correlation coefficient is given by

$$\rho_{RD_1,RD_2} = \frac{\sigma_{RD_1,RD_2}}{(\sigma^2_{RD_1}\sigma^2_{RD_2})^{\frac{1}{2}}} = -\frac{\sigma^2_{ADR_1}}{\sigma_{RD_1}\sigma_{RD_2}} \tag{F6}$$

Let $\sigma_i = \sigma_{RD_i}$ and $\rho_i = \rho_{RD_i,RD_{i+1}}$. Then the measurement noise covariance matrix, $P_{\nu'}$, for a sequence of m pairwise correlated range difference observations is given by

$$P_{\nu'} = \begin{pmatrix} \sigma^2_1 & \rho_1\sigma_1\sigma_2 & 0 & & & \\ \rho_1\sigma_1\sigma_2 & \sigma^2_2 & \rho_2\sigma_2\sigma_3 & & 0 & \\ 0 & \rho_2\sigma_2\sigma_3 & \sigma^2_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \sigma^2_{m-1} & \rho_{m-1}\sigma_{m-1}\sigma_m \\ & & & & \rho_{m-1}\sigma_{m-1}\sigma_m & \sigma^2_m \end{pmatrix} \tag{F7}$$

The corresponding linear measurement model for these m observations is given by

$$z' = A'\Delta x + \nu' \tag{F8}$$

where $E(\nu') = 0$ and $E(\nu' \nu'^T) = P_{\nu'}$. To whiten and decorrelate these observations, i.e., to transform the observations into an equivalent set of independent observations each with unit variance, do the following:

Factor $P_{\nu'}$ into the form LL^T where L is a lower triangular matrix. Then transform the linear measurement model given by equation (F8) by multiplying each side by L^{-1} to get

$$L^{-1}z' = L^{-1}A'\Delta x + L^{-1}\nu' \tag{F9}$$

or

$$z = A\Delta x + \nu \tag{F10}$$

where

$$E(\nu) = E(L^{-1}\nu') = L^{-1}E(\nu') = 0 \tag{F11}$$

and

$$\begin{aligned} E(\nu\nu^T) &= E(L^{-1}\nu'\nu'^T L^{-T}) \\ &= L^{-1}E(\nu'\nu'^T)L^{-T} \\ &= L^{-1}P_{\nu'}L^{-T} \\ &= L^{-1}LL^TL^{-T} \\ &= I \end{aligned} \tag{F12}$$

Therefore equation (F10) is the linear measurement model for the equivalent set of uncorrelated, unit variance observations required for the Filter measurement update step.

The banded tridiagonal matrix, $P_{\nu'}$, has a lower triangular square root, L, of the form

$$L = \begin{pmatrix} \bar{\sigma}_1 & & & & \\ \bar{\rho}_1 & \bar{\sigma}_2 & & & 0 \\ & \bar{\rho}_2 & \bar{\sigma}_3 & & \\ & & \ddots & \ddots & \\ 0 & & & \bar{\rho}_{m-1} & \bar{\sigma}_m \end{pmatrix} \tag{F13}$$

Then $P_{\nu'}$ is also given by

$$P_{v'} = LL^T = \begin{pmatrix} \bar{\sigma}_1^2 & \bar{\sigma}_1\bar{\beta}_1 & 0 & & & \\ \bar{\sigma}_1\bar{\beta}_1 & \bar{\sigma}_2^2+\bar{\beta}_1^2 & \bar{\sigma}_2\bar{\beta}_2 & & 0 & \\ 0 & \bar{\sigma}_2\bar{\beta}_2 & \bar{\sigma}_3^2+\bar{\beta}_2^2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \bar{\sigma}_{m-1}^2+\bar{\beta}_{m-2}^2 & \bar{\sigma}_{m-1}\bar{\beta}_{m-1} \\ & & & & \bar{\sigma}_{m-1}\bar{\beta}_{m-1} & \bar{\sigma}_m^2+\bar{\beta}_{m-1}^2 \end{pmatrix} \tag{F14}$$

Equating terms between the matrices given in equations (F14) and (F7) gives

$$\begin{aligned} \bar{\sigma}_1 &= \sigma_1 \\ \bar{\sigma}_1\bar{\beta}_1 &= \rho_1\sigma_1\sigma_2 \;\rightarrow\; \bar{\beta}_1 = \frac{\rho_1\sigma_1\sigma_2}{\bar{\sigma}_1} \\ \bar{\sigma}_2^2+\bar{\beta}_1^2 &= \sigma_2^2 \;\rightarrow\; \bar{\sigma}_2 = (\sigma_2^2-\bar{\beta}_1^2)^{\frac{1}{2}} \\ \bar{\sigma}_2\bar{\beta}_2 &= \rho_2\sigma_2\sigma_3 \;\rightarrow\; \bar{\beta}_2 = \frac{\rho_2\sigma_2\sigma_3}{\bar{\sigma}_2} \\ \bar{\sigma}_3^2+\bar{\beta}_2^2 &= \sigma_3^2 \;\rightarrow\; \bar{\sigma}_3 = (\sigma_3^2-\bar{\beta}_2^2)^{\frac{1}{2}} \end{aligned} \tag{F15}$$

or, in general,

$$\left.\begin{aligned} \bar{\beta}_n &= \frac{\rho_n\sigma_n\sigma_{n+1}}{\bar{\sigma}_n} \\ \bar{\sigma}_{n+1} &= (\sigma_{n+1}^2-\bar{\beta}_n^2)^{\frac{1}{2}} \end{aligned}\right\} \quad \begin{matrix} n = 1, 2, \ldots, m\text{-}1 \\ \bar{\sigma}_1 = \sigma_1 \end{matrix} \tag{F16}$$

The L^{-1} matrix is not actually computed but the transformation given by equation (F9) is done recursively as follows:

Let $\mathbf{c}'$ represent z', $\mathbf{v}'$, or any column of A'. Then $\mathbf{c}' = L\mathbf{c}$ is given by

$$\begin{pmatrix} c'_1 \\ c'_2 \\ c'_3 \\ \vdots \\ c'_m \end{pmatrix} = \begin{pmatrix} \bar{\sigma}_1 & & & & \\ \bar{\beta}_1 & \bar{\sigma}_2 & & 0 & \\ & \bar{\beta}_2 & \bar{\sigma}_3 & & \\ & & \ddots & \ddots & \\ 0 & & & \bar{\beta}_{m-1} & \bar{\sigma}_m \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_m \end{pmatrix} \tag{F17}$$

Individual equations are then given by

$$
\begin{aligned}
c'_1 &= \bar{\delta}_1 c_1 \\
c'_2 &= \bar{\beta}_1 c_1 + \bar{\delta}_2 c_2 \\
c'_3 &= \bar{\beta}_2 c_2 + \bar{\delta}_3 c_3 \\
&\vdots \\
c'_m &= \bar{\beta}_{m-1} c_{m-1} + \bar{\delta}_m c_m
\end{aligned}
\tag{F18}
$$

Solving those equations recursively for the c_i's then gives

$$
\begin{aligned}
c_1 &= c'_1 / \bar{\delta}_1 \\
c_2 &= (c'_2 - \bar{\beta}_1 c_1) / \bar{\delta}_2 \\
c_3 &= (c'_3 - \bar{\beta}_2 c_2) / \bar{\delta}_3 \\
&\vdots \\
c_m &= (c'_m - \bar{\beta}_{m-1} c_{m-1}) / \bar{\delta}_m
\end{aligned}
\tag{F19}
$$

APPENDIX G

HOUSEHOLDER ORTHOGONAL TRANSFORMATIONS

This appendix is not intended to provide a detailed explanation of Householder transformations and their properties. The reader is referred to Chapter IV of reference 4 for this explanation. However, two properties of these transformations exploited in the MSF/S implementation will be discussed below. First it will be shown that the transformation defined by equations (105-110) in the MEASUREMENT UPDATE subsection of the FILTER ALGORITHM section does zero out the below diagonal elements in the first column of the arbitrary $m \times n$ matrix R.

For the first column, equation (109) gives

$$\gamma_1 = \frac{1}{su(1)} \begin{pmatrix} R(1,1)-s \\ R(2,1) \\ \vdots \\ R(m,1) \end{pmatrix}^T R(i,1) = \frac{1}{su(1)} \left(|R(i,1)|^2 - sR(1,1) \right) \tag{G1}$$

Squaring equation (105) gives

$$s^2 = |R(i,1)|^2 \tag{G2}$$

Substituting equation (G2) in (G1) gives

$$\gamma_1 = \frac{1}{su(1)} \left(s^2 - sR(1,1) \right) = \frac{1}{u(1)} \left(s - R(1,1) \right) \tag{G3}$$

Substituting in equation (G3) for $u(1)$ from equation (106) gives

$$\gamma_1 = \frac{s - R(1,1)}{R(1,1) - s} = -1 \tag{G4}$$

Therefore for $j = 1$, equation (110) gives

$$T_u(R(i,1)) = R(i,1) - \begin{pmatrix} R(1,1)-s \\ R(2,1) \\ \vdots \\ R(m,1) \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{G5}$$

Two properties of Householder orthogonal transformations are exploited to reduce the number of computations in the MSF/S implementation. These are given as follows:

1. If the current column being zeroed out below the diagonal has a zero element, the corresponding row for all remaining columns is unchanged by this transformation. $u(i) = 0$ for this row in equation (110) so that $R(i,j)$ for all columns j does not change. This property allows the Householder transformations required for the measurement update to be done in two stages (equations (103) and (104)) and saves

over $N_y \times (N_p + N_x)$ words of array storage. Since the R_y array is always upper triangular, this property allows it to be stored and operated on as a one-dimensional array in equation (104). The Filter propagation step computations are also reduced by applying this property. In equation (111) the matrix $-R_w M$ is always upper triangular so that only the highest row being operated on in the first N_p rows changes as each column is zeroed below the diagonal.

2. If any column being transformed has zero entries corresponding to all nonzero entries in the current column being zeroed out below the diagonal, this entire column is unchanged by this transformation. $\gamma_j = 0$ for this column in equation (110) so that $R(i,j)$ for all rows i does not change. This property is applied as described under equation (111) in the **PROPAGATION** subsection of the **FILTER ALGORITHM** section. The Gauss-Markov p parameters were placed first in the list of stochastic parameters in order to take maximum advantage of this property.

APPENDIX H

SRIF AND EQUIVALENCE TO KALMAN FILTER

The square root information filter (SRIF) measurement update equations (equations (103) and (104)) corresponding to the parameter set partitioning implemented in the MSF/S system are derived in reference 4, section VII.2 and are given there by equations (2.6) and (2.7). The SRIF propagation equations (equation (111)) are derived in reference 4, section VII.3 and are given there in a more general form by equation (3.6) repeated here as

$$\tilde{T}_p \begin{pmatrix} -R_w M & R_w & 0 & 0 & z_w \\ \hat{R}_p - \bar{R}_{px} V_p & 0 & \bar{R}_{px} & \hat{R}_{py} - \bar{R}_{px} V_y & \hat{z}_p \\ \hat{R}_{xp} - \bar{R}_x V_p & 0 & \bar{R}_x & \hat{R}_{xy} - \bar{R}_x V_y & \hat{z}_x \end{pmatrix} = \begin{pmatrix} R_p^* & R_{pp}^* & R_{px}^* & R_{py}^* & z_p^* \\ 0 & \tilde{R}_p & \tilde{R}_{px} & \tilde{R}_{py} & \tilde{z}_p \\ 0 & \tilde{R}_{xp} & \tilde{R}_x & \tilde{R}_{xy} & \tilde{z}_x \end{pmatrix} \tag{H1}$$

$$\text{where } \bar{R}_{px} = \hat{R}_{px} V_x^{-1} \text{ and } \bar{R}_x = \hat{R}_x V_x^{-1} \tag{H2}$$

These equations reduce to that used in the MSF/S system with the following simplifications:

$$z_w = 0 \tag{H3}$$

i.e., no *a priori* knowledge of process noise exists except for its covariance matrix,

$$V_y = 0 \tag{H4}$$

$$\text{and } V_x = I \tag{H5}$$

because of the pseudoepoch state variable definitions (see Appendices A and D), and

$$\hat{R}_{xp} = 0 \tag{H6}$$

because the measurement update Householder transformation is applied even if no observations are present in a given mini-batch interval.

An alternate derivation of the propagation equations can be obtained by substituting partitioned ϕ and G matrices into the general propagation algorithm given by equation (2.29) in reference 4, section VI.2 and rearranging the result as follows:

The general form of the state equations is given by equation (1) and repeated here as

$$\Delta \boldsymbol{x}_{j+1} = \phi_j \Delta \boldsymbol{x}_j + G \boldsymbol{w}_j \tag{H7}$$

The MSF/S state equations correspond to this form under the following definitions:

$$\Delta \boldsymbol{x}_j = \begin{pmatrix} \Delta p \\ \Delta x \\ \Delta y \end{pmatrix}_j \tag{H8}$$

$$\Phi_j = \begin{array}{cl} \begin{pmatrix} M & 0 & 0 \\ V_{p_j} & I & 0 \\ 0 & 0 & I \end{pmatrix} & \begin{array}{l} \} \; N_p \\ \} \; N_x \\ \} \; N_y \end{array} \end{array} \tag{H9}$$

$$G = \begin{array}{cl} \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} & \begin{array}{l} \} \; N_p \\ \\ \\ \end{array} \end{array} \tag{H10}$$

The general form of the information array being transformed in the propagation step is given by the following, taken from reference 4, equation (2.29), with $z_w(j) = 0$:

$$\begin{pmatrix} R_w(j) & 0 & 0 \\ -\hat{R}_j \Phi_j^{-1} G & \hat{R}_j \Phi_j^{-1} & \hat{z}_j \end{pmatrix} \tag{H11}$$

This is equivalent to the data equations given by

$$R_w(j) w_j = -v_{w_j} \tag{H12}$$

$$-\hat{R}_j \Phi_j^{-1} G w_j + \hat{R}_j \Phi_j^{-1} \Delta x_j = \hat{z}_j - \hat{v}_j \tag{H13}$$

For the partitioning done in the MSF/S system $\hat{R}_j$ and $\hat{z}_j$ are given by

$$\hat{R}_j = \begin{pmatrix} \hat{R}_p & \hat{R}_{px} & \hat{R}_{py} \\ 0 & \hat{R}_x & \hat{R}_{xy} \\ 0 & 0 & \hat{R}_y \end{pmatrix}_j \tag{H14}$$

$$\hat{z}_j = \begin{pmatrix} \hat{z}_p \\ \hat{z}_x \\ \hat{z}_y \end{pmatrix}_j \tag{H15}$$

Inverting both sides of equation (H9) gives

$$\Phi_j^{-1} = \begin{pmatrix} M^{-1} & 0 & 0 \\ -V_{p_j} M^{-1} & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{H16}$$

Then $\hat{R}_j \Phi_j^{-1}$ and $-\hat{R}_j \Phi_j^{-1} G$ are given by

$$\hat{R}_j \Phi_j^{-1} = \begin{pmatrix} \hat{R}_p M^{-1} - \hat{R}_{px} V_p M^{-1} & \hat{R}_{px} & \hat{R}_{py} \\ -\hat{R}_x V_p M^{-1} & \hat{R}_x & \hat{R}_{xy} \\ 0 & 0 & \hat{R}_y \end{pmatrix}_j \tag{H17}$$

$$-\hat{R}_j\Phi_j^{-1}G = \begin{pmatrix} -(\hat{R}_pM^{-1} - \hat{R}_{px}V_pM^{-1}) \\ \hat{R}_xV_pM^{-1} \\ 0 \end{pmatrix}_j \tag{H18}$$

Substituting equations (H15), (H17), and (H18) into equation (H13) gives

$$-(\hat{R}_{p_j}M^{-1} - \hat{R}_{px_j}V_{p_j}M^{-1})\mathbf{w}_j + (\hat{R}_{p_j}M^{-1} - \hat{R}_{px_j}V_{p_j}M^{-1})\Delta\mathbf{p}_{j+1} + \hat{R}_{px_j}\Delta\mathbf{x}_{j+1} + \hat{R}_{py_j}\Delta\mathbf{y}_{j+1} = \hat{\mathbf{z}}_{p_j} - \hat{\mathbf{v}}_{p_j} \tag{H19}$$

$$\hat{R}_{x_j}V_{p_j}M^{-1}\mathbf{w}_j - \hat{R}_{x_j}V_{p_j}M^{-1}\Delta\mathbf{p}_{j+1} + \hat{R}_{x_j}\Delta\mathbf{x}_{j+1} + \hat{R}_{xy_j}\Delta\mathbf{y}_{j+1} = \hat{\mathbf{z}}_{x_j} - \hat{\mathbf{v}}_{x_j} \tag{H20}$$

$$\hat{R}_{y_j}\Delta\mathbf{y}_{j+1} = \hat{\mathbf{z}}_{y_j} - \hat{\mathbf{v}}_{y_j} \tag{H21}$$

Solving the state equations for $\mathbf{w}_j$ gives

$$\mathbf{w}_j = -M\Delta\mathbf{p}_j + \Delta\mathbf{p}_{j+1} \tag{H22}$$

Multiplying each side of equation (H22) by $R_w(j)$ gives

$$R_w(j)\mathbf{w}_j = -R_w(j)M\Delta\mathbf{p}_j + R_w(j)\Delta\mathbf{p}_{j+1} \tag{H23}$$

Ignoring equation (H21), since y parameters do not change in the propagation step, and substituting equation (H23) into equation (H12) and equation (H22) into equations (H19) and (H20) gives

$$-R_w(j)M\Delta\mathbf{p}_j + R_w(j)\Delta\mathbf{p}_{j+1} = -\mathbf{v}_{w_j} \tag{H24}$$

$$(\hat{R}_{p_j} - \hat{R}_{px_j}V_{p_j})\Delta\mathbf{p}_j + \hat{R}_{px_j}\Delta\mathbf{x}_{j+1} + \hat{R}_{py_j}\Delta\mathbf{y}_{j+1} = \hat{\mathbf{z}}_{p_j} - \hat{\mathbf{v}}_{p_j} \tag{H25}$$

$$-\hat{R}_{x_j}V_{p_j}\Delta\mathbf{p}_j + \hat{R}_{x_j}\Delta\mathbf{x}_{j+1} + \hat{R}_{xy_j}\Delta\mathbf{y}_{j+1} = \hat{\mathbf{z}}_{x_j} - \hat{\mathbf{v}}_{x_j} \tag{H26}$$

This is a set of data equations for the states $\Delta\mathbf{p}_j$, $\Delta\mathbf{p}_{j+1}$, $\Delta\mathbf{x}_{j+1}$, and $\Delta\mathbf{y}_{j+1}$, and corresponds to the information array required for the propagation step in the MSF/S implementation given by

$$\begin{pmatrix} -R_wM & R_w & 0 & 0 & 0 \\ \hat{R}_p - \hat{R}_{px}V_p & 0 & \hat{R}_{px} & \hat{R}_{py} & \hat{\mathbf{z}}_p \\ -\hat{R}_xV_p & 0 & \hat{R}_x & \hat{R}_{xy} & \hat{\mathbf{z}}_x \end{pmatrix}_j \tag{H27}$$

The Householder transformation, $\hat{T}_p$, operating on this matrix eliminates $\Delta\mathbf{p}_j$ from the last $N_p + N_x$ rows.

The SRIF algorithm is mathematically equivalent to the standard Kalman filter algorithm. Assume a set of state equations given by equation (1) (repeated above as equation (H7)) and a linear measurement model given by equation (2) and repeated here as

$$\mathbf{z}_j = A_j\Delta\mathbf{x}_j + \mathbf{v}_j \tag{H28}$$

Then this equivalence will be shown for each filter processing step–measurement update and propagation.

Measurement Update

The Kalman filter measurement update equations are given by

$$\hat{\Delta \mathbf{x}}_j = \tilde{\Delta \mathbf{x}}_j + K_j(\mathbf{z}_j - A_j \tilde{\Delta \mathbf{x}}_j) \tag{H29}$$

$$\hat{P}_j = (I - K_j A_j)\tilde{P}_j = (\tilde{P}_j^{-1} + A_j^T A_j)^{-1} \tag{H30}$$

$$K_j = \tilde{P}_j A_j^T (A_j \tilde{P}_j A_j^T + I)^{-1} = \hat{P}_j A_j^T \tag{H31}$$

where $\sim$ indicates a predicted quantity and $\wedge$ indicates a filter estimated quantity. A different expression for $\Delta \mathbf{x}_j$ can be derived by substituting for K_j from equation (H31) into equation (H29) and using equation (H30) to give

$$\begin{aligned} \hat{\Delta \mathbf{x}}_j &= \tilde{\Delta \mathbf{x}}_j + \hat{P}_j A_j^T (\mathbf{z}_j - A_j \tilde{\Delta \mathbf{x}}_j) \\ &= (I - \hat{P}_j A_j^T A_j)\tilde{\Delta \mathbf{x}}_j + \hat{P}_j A_j^T \mathbf{z}_j \\ &= \hat{P}_j[(\hat{P}_j^{-1} - A_j^T A_j)\tilde{\Delta \mathbf{x}}_j + A_j^T \mathbf{z}_j] \\ &= \hat{P}_j(\tilde{P}_j^{-1}\tilde{\Delta \mathbf{x}}_j + A_j^T \mathbf{z}_j) \end{aligned} \tag{H32}$$

The SRIF measurement update equations are given in non-partitioned form by

$$\hat{T}\begin{pmatrix} \tilde{R}_j & \tilde{\mathbf{z}}_j \\ A_j & \mathbf{z}_j \end{pmatrix} = \begin{pmatrix} \hat{R}_j & \hat{\mathbf{z}}_j \\ 0 & \mathbf{e}_j \end{pmatrix} \tag{H33}$$

Premultiply each side by its transpose to get

$$\begin{pmatrix} \tilde{R}_j^T & A_j^T \\ \tilde{\mathbf{z}}_j^T & \mathbf{z}_j^T \end{pmatrix} \hat{T}^T\hat{T} \begin{pmatrix} \tilde{R}_j & \tilde{\mathbf{z}}_j \\ A_j & \mathbf{z}_j \end{pmatrix} = \begin{pmatrix} \hat{R}_j^T & 0 \\ \hat{\mathbf{z}}_j^T & \mathbf{e}_j^T \end{pmatrix} \begin{pmatrix} \hat{R}_j & \hat{\mathbf{z}}_j \\ 0 & \mathbf{e}_j \end{pmatrix} \tag{H34}$$

Since $\hat{T}$ is orthogonal, $\hat{T}^T\hat{T} = I$ so equation (H34) reduces to

$$\begin{pmatrix} \tilde{R}_j^T\tilde{R}_j + A_j^T A_j & \tilde{R}_j^T\tilde{\mathbf{z}}_j + A_j^T\mathbf{z}_j \\ \tilde{\mathbf{z}}_j^T\tilde{R}_j + \mathbf{z}_j^T A_j & \tilde{\mathbf{z}}_j^T\tilde{\mathbf{z}}_j + \mathbf{z}_j^T\mathbf{z}_j \end{pmatrix} = \begin{pmatrix} \hat{R}_j^T\hat{R}_j & \hat{R}_j^T\hat{\mathbf{z}}_j \\ \hat{\mathbf{z}}_j^T\hat{R}_j & \hat{\mathbf{z}}_j^T\hat{\mathbf{z}}_j + \mathbf{e}_j^T\mathbf{e}_j \end{pmatrix} \tag{H35}$$

Equating upper left terms gives

$$\tilde{R}_j^T\tilde{R}_j + A_j^T A_j = \hat{R}_j^T\hat{R}_j \tag{H36}$$

Inverting both sides gives

$$\hat{R}_j^{-1}\hat{R}_j^{-T} = (\tilde{R}_j^T\tilde{R}_j + A_j^T A_j)^{-1} \tag{H37}$$

Subtituting $P = R^{-1}R^{-T}$ in equation (H37) then gives

$$\hat{P}_j = (\tilde{P}_j^{-1} + A_j^T A_j)^{-1} \tag{H38}$$

which is the same as equation (H30) above.

Equating upper right terms in equation (H35) gives

$$\tilde{R}_j^T \tilde{z}_j + A_j^T z_j = \hat{R}_j^T \hat{z}_j \tag{H39}$$

Mulitplying $z = R\Delta x$ by R^T on each side gives

$$R^T z = R^T R \Delta x \tag{H40}$$

Substituting equation (H40) into equation (H39) for terms involving ~ and ^ gives

$$\tilde{R}_j^T \tilde{R}_j \tilde{\Delta x}_j + A_j^T z_j = \hat{R}_j^T \hat{R}_j \hat{\Delta x}_j \tag{H41}$$

Since $P = R^{-1}R^{-T}$, $P^{-1} = R^T R$ so that equation (H41) becomes

$$\tilde{P}_j^{-1} \tilde{\Delta x}_j + A_j^T z_j = \hat{P}_j^{-1} \hat{\Delta x}_j \tag{H42}$$

Multiplying each side by $\hat{P}_j$ and regrouping terms then gives

$$\hat{\Delta x}_j = \hat{P}_j \, (\tilde{P}_j^{-1} \tilde{\Delta x}_j + A^T z_j) \tag{H43}$$

which is the same as equation (H32) above.

Propagation

The Kalman filter propagation equations are given by

$$\tilde{\Delta x}_{j+1} = \phi_j \hat{\Delta x}_j \tag{H44}$$

$$\tilde{P}_{j+1} = \phi_j \hat{P}_j \phi_j^T + G Q_j G^T \tag{H45}$$

The SRIF propagation equations are given by

$$\tilde{T} \begin{array}{c} \begin{array}{ccc} \underbrace{N_w} & \underbrace{N_x} & 1 \end{array} \\ \begin{pmatrix} R_w(j) & 0 & 0 \\ -\hat{R}_j \phi_j^{-1} G & \hat{R}_j \phi_j^{-1} & \hat{z}_j \end{pmatrix} \end{array} = \begin{pmatrix} \tilde{R}_w(j) & \tilde{R}_{wx}(j) & \tilde{z}_w(j) \\ 0 & \tilde{R}_{j+1} & \tilde{z}_{j+1} \end{pmatrix} \begin{array}{l} \} \, N_w \\ \} \, N_x \end{array} \tag{H46}$$

Partition $\tilde{T}$ as follows:

$$\tilde{T} = \begin{array}{c} \begin{array}{cc} \underbrace{N_w} & \underbrace{N_x} \end{array} \\ \begin{pmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{pmatrix} \end{array} \begin{array}{l} \} \, N_w \\ \} \, N_x \end{array} \tag{H47}$$

Using these definitions it follows that

$$\tilde{R}_{j+1} = \tilde{T}_{22}\hat{R}_j\Phi_j^{-1} \quad \text{or} \quad \tilde{T}_{22} = \tilde{R}_{j+1}\Phi_j\hat{R}_j^{-1} \tag{H48}$$

$$\begin{aligned}&\tilde{T}_{21}R_w(j) - \tilde{T}_{22}\hat{R}_j\Phi_j^{-1}G = 0 \qquad \text{or}\\ &\qquad \tilde{T}_{21} = \tilde{T}_{22}\hat{R}_j\Phi_j^{-1}G\,R_w^{-1}(j) = \tilde{R}_{j+1}G\,R_w^{-1}(j)\end{aligned} \tag{H49}$$

Also, since $\tilde{T}$ is orthogonal it follows that

$$\tilde{T}_{21}\tilde{T}_{21}^T + \tilde{T}_{22}\tilde{T}_{22}^T = I \tag{H50}$$

Substituting for $\tilde{T}_{22}$ and $\tilde{T}_{21}$ from equations (H48) and (H49) into equation (H50) gives

$$I = \tilde{R}_{j+1}G\,R_w^{-1}(j)\,R_w^{-T}(j)\,G^T\tilde{R}_{j+1}^T + \tilde{R}_{j+1}\Phi_j\,\hat{R}_j^{-1}\,\hat{R}_j^{-T}\Phi_j^T\tilde{R}_{j+1}^T \tag{H51}$$

Pre-multiplying by $\tilde{R}_{j+1}^{-1}$ and post-multiplying by $\tilde{R}_{j+1}^{-T}$ gives

$$\tilde{R}_{j+1}^{-1}\tilde{R}_{j+1}^{-T} = G\,R_w^{-1}(j)\,R_w^{-1}(j)\,G^T + \Phi_j\,\hat{R}_j^{-1}\,\hat{R}_j^{-T}\Phi_j^T \tag{H52}$$

Substituting $P = R^{-1}R^{-T}$ and $Q_j = \; R_w^{-1}(j)\,R_w^{-T}(j)$ then gives

$$\tilde{P}_{j+1} = \Phi_j\hat{P}_j\Phi_j^T + GQ_jG^T \tag{H53}$$

which is the same as equation (H45) above.

Also the definitions in equations (H46) and (H47) give

$$\tilde{T}_{22}\hat{\mathbf{z}}_j = \tilde{\mathbf{z}}_j \tag{H54}$$

Substituting for $\tilde{T}_{22}$ from equ tion (H48) above gives

$$\tilde{R}_{j+1}\Phi_j\,\hat{R}_j^{-1}\hat{\mathbf{z}}_j = \tilde{\mathbf{z}}_{j+1} \tag{H55}$$

Since $\mathbf{z} = R\Delta\mathbf{x}$, this becomes

$$\tilde{R}_{j+1}\Phi_j\hat{R}_j^{-1}\hat{R}_j\hat{\Delta\mathbf{x}}_j = \tilde{R}_{j+1}\Phi_j\hat{\Delta\mathbf{x}}_j = \tilde{R}_{j+1}\tilde{\Delta\mathbf{x}}_{j+1} \tag{H56}$$

Multiplying both sides by $\tilde{R}_{j+1}^{-1}$ then gives

$$\Phi_j\hat{\Delta\mathbf{x}}_j = \tilde{\Delta\mathbf{x}}_{j+1} \tag{H57}$$

which is the same as equation (H44) above.

APPENDIX I

SRIS AND EQUIVALENCE TO RTS SMOOTHER

The square root information smoother (SRIS) is implemented using two algorithms in the MSF/S system. The first algorithm, called the State Only Smoother, provides state estimates only. The second algorithm, called the Array Smoother, in addition provides covariance estimates. Both smoothing algorithms are developed in reference 4, section X.2 for the general form of the state equations given by equation (H7). For the special form of the state equations defined by equations (H8), (H9), and (H10), the Array Smoother and State Only Smoother can be derived by the proper substitutions as follows:

The information array, corresponding to the general form of the state equations, that is transformed in the SRIS algorithm is given in reference 4, section X.2, equation 2.7 and is repeated here as

$$\begin{pmatrix} \tilde{R}_w(j) + \tilde{R}_{wx}(j)G & \tilde{R}_{wx}(j)\Phi_j & \tilde{z}_{w_j} \\ R_x^*(t_{j+1})G & R_x^*(t_{j+1})\Phi_j & z_x^*(t_{j+1}) \end{pmatrix} \tag{I1}$$

The data equation saved for smoothing in the general case based on reference 4, section VI.2, equation 2.29 is given by

$$\tilde{R}_w(j)\mathbf{w}_j + \tilde{R}_{wx}(j)\Delta\mathbf{x}_{j+1} = \tilde{z}_{w_j} - \tilde{\nu}_{w_j} \tag{I2}$$

The corresponding data equation saved for smoothing in the MSF/S propagation step is given by

$$\tilde{R}_p(t_j)\Delta\mathbf{p}_j + \tilde{R}_{pp}(t_j)\Delta\mathbf{p}_{j+1} + \tilde{R}_{px}(t_j)\Delta\mathbf{x}_{j+1} + \tilde{R}_{py}(t_j)\Delta\mathbf{y}_N = \tilde{z}_{p_j} - \tilde{\nu}_{p_j} \tag{I3}$$

From the state equations it follows that

$$\Delta\mathbf{p}_{j+1} = M\Delta\mathbf{p}_j + \mathbf{w}_j \text{ or } \Delta\mathbf{p}_j = M^{-1}\Delta\mathbf{p}_{j+1} - M^{-1}\mathbf{w}_j \tag{I4}$$

Substituting for $\Delta\mathbf{p}_j$ in equation (I3) above gives

$$-\tilde{R}_p(t_j)M^{-1}\mathbf{w}_j + (\tilde{R}_p(t_j)M^{-1} + \tilde{R}_{pp}(t_j))\Delta\mathbf{p}_{j+1} + \tilde{R}_{px}(t_j)\Delta\mathbf{x}_{j+1} + \tilde{R}_{py}(t_j)\Delta\mathbf{y}_N = \tilde{z}_{p_j} - \tilde{\nu}_{p_j} \tag{I5}$$

Comparing terms between equations (I5) and (I2) gives

$$\tilde{R}_w(j) = -\tilde{R}_p(t_j)M^{-1} \tag{I6}$$

$$\tilde{R}_{wx}(j) = (\tilde{R}_p(t_j)M^{-1} + \tilde{R}_{pp}(t_j) \quad \tilde{R}_{px}(t_j) \quad \tilde{R}_{py}(t_j)) \tag{I7}$$

Therefore

$$\tilde{R}_w(j) + \tilde{R}_{wx}(j)G = -\tilde{R}_p(t_j)M^{-1} + \tilde{R}_p(t_j)M^{-1} + \tilde{R}_{pp}(t_j) = \tilde{R}_{pp}(t_j) \tag{I8}$$

$$\tilde{R}_{wx}(j)\Phi_j = (\tilde{R}_p(t_j) + \tilde{R}_{pp}(t_j)M + \tilde{R}_{px}(t_j)V_{p_j} \quad \tilde{R}_{px}(t_j) \quad \tilde{R}_{py}(t_j)) \tag{I9}$$

$$\tilde{z}_{w_j} = \tilde{z}_{p_j} \tag{I10}$$

Also

$$R_x^*(t_{j+1})G = \begin{pmatrix} R_p^*(t_{j+1}) & R_{px}^*(t_{j+1}) & R_{py}^*(t_{j+1}) \\ 0 & R_x^*(t_{j+1}) & R_{xy}^*(t_{j+1}) \\ 0 & 0 & R_y^*(t_{j+1}) \end{pmatrix} \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_p^*(t_{j+1}) \\ 0 \\ 0 \end{pmatrix} \qquad \text{(I11)}$$

$$\begin{aligned} R_x^*(t_{j+1})\Phi_j &= \begin{pmatrix} R_p^*(t_{j+1}) & R_{px}^*(t_{j+1}) & R_{py}^*(t_{j+1}) \\ 0 & R_x^*(t_{j+1}) & R_{xy}^*(t_{j+1}) \\ 0 & 0 & R_y^*(t_{j+1}) \end{pmatrix} \begin{pmatrix} M & 0 & 0 \\ V_{p_j} & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} R_p^*(t_{j+1})M + R_{px}^*(t_{j+1})V_{p_j} & R_{px}^*(t_{j+1}) & R_{py}^*(t_{j+1}) \\ R_x^*(t_{j+1})V_{p_j} & R_x^*(t_{j+1}) & R_{xy}^*(t_{j+1}) \\ 0 & 0 & R_y^*(t_{j+1}) \end{pmatrix} \end{aligned} \qquad \text{(I12)}$$

and

$$z_x^*(t_{j+1}) = \begin{pmatrix} z_p^*(t_{j+1}) \\ z_x^*(t_{j+1}) \\ z_y^*(t_{j+1}) \end{pmatrix} \qquad \text{(I13)}$$

Substituting equations (I8) through (I13) into equation (I1) gives the smoothing array for the MSF/S implementation as

$$\begin{pmatrix} \tilde{R}_{pp}(t_j) & \tilde{R}_p(t_j) + \tilde{R}_{pp}(t_j)M + \tilde{R}_{px}(t_j)V_{p_j} & \tilde{R}_{px}(t_j) & \tilde{R}_{py}(t_j) & \tilde{z}_{p_j} \\ R_p^*(t_{j+1}) & R_p^*(t_{j+1})M + R_{px}^*(t_{j+1})V_{p_j} & R_{px}^*(t_{j+1}) & R_{py}^*(t_{j+1}) & z_p^*(t_{j+1}) \\ 0 & R_x^*(t_{j+1})V_{p_j} & R_x^*(t_{j+1}) & R_{xy}^*(t_{j+1}) & z_x^*(t_{j+1}) \\ 0 & 0 & 0 & R_y^*(t_{j+1}) & z_y^*(t_{j+1}) \end{pmatrix} \qquad \text{(I14)}$$

Also $R_y^*(t_j) = \tilde{R}_y(t_N)$ and $z_y^*(t_j) = \tilde{z}_y(t_N)$ for all $j = 0, 1, \ldots, N$ where $\tilde{R}_y(t_N)$ is upper triangular. These last N_y rows are therefore not carried in the smoothing computations. The matrices $\tilde{R}_p(t_j)$, $R_p^*(t_{j+1})$, and $R_x^*(t_{j+1})$ are always upper triangular. The Householder transformation, T_{ppx}^*, operating on this matrix eliminates Δp_{j+1}, from the last $N_p + N_x + N_y$ rows.

The State Only Smoother algorithm corresponding to the general form of the state equations is given in reference 4, section X.2, equations (2.1) and (2.2) and is repeated he:·› as

$$\Delta x_N^* = \hat{R}_N^{-1}\hat{z}_N \qquad \text{(I15)}$$

$$\left.\begin{aligned} w_j^* &= [\tilde{R}_w(j)]^{-1}(\tilde{z}_w(j) - \tilde{R}_{wx}(j)\Delta x_{j+1}^*) \\ \Delta x_j^* &= \Phi_j^{-1}(\Delta x_{j+1}^* - G w_j^*) \end{aligned}\right\} \quad j = N-1, N-2, \ldots, 0 \qquad \text{(I16)}$$

Solving equation (I3) for Δp_j^* by setting $\tilde{v}_{p_j} = 0$ gives

$$\Delta p_j^* = [\tilde{R}_p(t_j)]^{-1}[\tilde{z}_{p_j} - \tilde{R}_{pp}(t_j)\Delta p_{j+1}^* - \tilde{R}_{px}(t_j)\Delta x_{j+1}^* - \tilde{R}_{py}(t_j)\Delta y_N^*] \tag{I17}$$

This is the same as equation (114) in the **STATE ONLY SMOOTHER** subsection of the **SMOOTHER ALGORITHMS** section. To derive equation (115) substitute the partitioned definitions of Δx, ϕ_j^{-1}, and G into the second part of equation (I16) to get

$$\begin{pmatrix} \Delta p^* \\ \Delta x^* \\ \Delta y^* \end{pmatrix}_j = \begin{pmatrix} M^{-1} & 0 & 0 \\ -V_{p_j}M^{-1} & I & 0 \\ 0 & 0 & I \end{pmatrix} \left[\begin{pmatrix} \Delta p^* \\ \Delta x^* \\ \Delta y^* \end{pmatrix}_{j+1} - \begin{pmatrix} w_j^* \\ 0 \\ 0 \end{pmatrix} \right] \tag{I18}$$

Extracting out the portion of equation (I18) involving Δx_j^* gives

$$\Delta x_j^* = -V_{p_j}M^{-1}(\Delta p_{j+1}^* - w_j^*) + \Delta x_{j+1}^* \tag{I19}$$

From the state equations

$$\Delta p_{j+1}^* = M\Delta p_j^* + w_j^* \text{ or } w_j^* = \Delta p_{j+1}^* - M\Delta p_j^* \tag{I20}$$

Substituting w_j^* into equation (I19) then gives

$$\begin{aligned} \Delta x_j^* &= -V_{p_j}M^{-1}(M\Delta p_j^*) + \Delta x_{j+1}^* \\ &= \Delta x_{j+1}^* - V_{p_j}\Delta p_j^* \end{aligned} \tag{I21}$$

This is the same as equation (115). Therefore equations (I15), (I17), and (I21) give the State Only Smoother algorithm in the MSF/S system.

The SRIS algorithm is mathematically equivalent to the standard Rauch-Tung-Striebel (RTS) smoothing algorithm. Assume a set of state equations of the general form given by equation (H7). Then this equivalence can be shown by proper manipulation of the smoother information arrays as follows:

The RTS smoother equations are given by

$$\Delta x_j^* = \hat{\Delta x}_j + C_j(\Delta x_{j+1}^* - \tilde{\Delta x}_{j+1}) \tag{I22}$$

$$P_j^* = \hat{P}_j + C_j(P_{j+1}^* - \tilde{P}_{j+1})C_j^T \tag{I23}$$

where $$C_j = \hat{P}_j \phi_j^T \tilde{P}_{j+1}^{-1} \tag{I24}$$

The SRIS equations are given by

$$T^* \begin{pmatrix} \tilde{R}_w(j) + \tilde{R}_{wx}(j)G & \tilde{R}_{wx}(j)\phi_j & \tilde{z}_w(j) \\ R_{j+1}^* G & R_{j+1}^*\phi_j & z_{j+1}^* \end{pmatrix} = \begin{pmatrix} R_w^*(j) & R_{wx}^*(j) & z_w^*(j) \\ 0 & R_j^* & z_j^* \end{pmatrix} \tag{I25}$$

Partition T^* as follows:

$$T^* = \begin{array}{cc} \overbrace{\phantom{T^*_{11}}}^{N_w} & \overbrace{\phantom{T^*_{12}}}^{N_x} \end{array}$$
$$T^* = \begin{pmatrix} T^*_{11} & T^*_{12} \\ T^*_{21} & T^*_{22} \end{pmatrix} \begin{array}{l} \} N_w \\ \} N_x \end{array} \tag{I26}$$

Using these definitions it follows that

$$0 = T^*_{21}(\tilde{R}_w(j) + \tilde{R}_{wx}(j)G) + T^*_{22}R^*_{j+1}G \tag{I27}$$

$$R^*_j = T^*_{21}\tilde{R}_{wx}(j)\Phi_j + T^*_{22}R^*_{j+1}\Phi_j \tag{I28}$$

Post-multiplying equation (I28) by $\Phi_j^{-1}G$ gives

$$R^*_j\Phi_j^{-1}G = T^*_{21}\tilde{R}_{wx}(j)G + T^*_{22}R^*_{j+1}G \tag{I29}$$

Subtracting equation (I27) from equation (I29) gives

$$R^*_j\Phi_j^{-1}G = -T^*_{21}\tilde{R}_w(j) \quad \text{or} \quad T^*_{21} = -R^*_j\Phi_j^{-1}G\,\tilde{R}_w^{-1}(j) \tag{I30}$$

Post-multiplying equation (I28) by Φ_j^{-1} and substituting for T^*_{21} from equation (I30) gives

$$R^*_j\Phi_j^{-1} = -R^*_j\Phi_j^{-1}G\,\tilde{R}_w^{-1}(j)\tilde{R}_{wx}(j) + T^*_{22}R^*_{j+1} \tag{I31}$$

Solving this for T^*_{22} gives

$$T^*_{22} = R^*_j\Phi_j^{-1}(I + G\tilde{R}_w^{-1}(j)\tilde{R}_{wx}(j))\,R^{*-1}_{j+1} \tag{I32}$$

In reference 4, section X.A it is shown from the propagation equations that

$$\Phi_j^{-1}(I + G\tilde{R}_w^{-1}(j)\tilde{R}_{wx}(j)) = \hat{P}_j\Phi_j^{T}\tilde{P}^{-1}_{j+1} = C_j \tag{I33}$$

Therefore

$$T^*_{22} = R^*_j C_j R^{*-1}_{j+1} \tag{I34}$$

Since T^* is orthogonal it follows that

$$T^*_{21}T^{*T}_{21} + T^*_{22}T^{*T}_{22} = I \tag{I35}$$

Substituting for T^*_{21} and T^*_{22} from equations (I30) and (I34) gives

$$I = R^*_j\Phi_j^{-1}G\,\tilde{R}_w^{-1}(j)\,\tilde{R}_w^{-T}(j)\,G^T\Phi_j^{-T}R^{*T}_j + R^*_jC_j\,R^{*-1}_{j+1}R^{*-T}_{j+1}C_j^TR^{*T}_j \tag{I36}$$

Pre-multiplying by R^{*-1}_j and post-multiplying by R^{*-T}_j and replacing $\tilde{R}_w^{-1}(j)\,\tilde{R}_w^{-T}(j)$ by $\tilde{Q}_j$, $R^{*-1}_{j+1}R^{*-T}_{j+1}$ by P^*_{j+1}, and $R^{*-1}_j\,R^{*-T}_j$ by P^*_j gives

$$P^*_j = \Phi_jG\tilde{Q}_jG^T\Phi_j^{-T} + C_jP^*_{j+1}C_j^T \tag{I37}$$

In reference 4, section X.A it is also shown that

$$\Phi_j^{-1} G \tilde{Q}_j G^T \Phi_j^{-T} = \hat{P}_j - \hat{P}_j \Phi_j^T \tilde{P}_{j+1}^{-1} \Phi_j \hat{P}_j \tag{I38}$$

Using the second part of equation (I33) in equation (I38) gives

$$\begin{aligned} \Phi_j^{-1} G \tilde{Q}_j G^T \Phi_j^T &= \hat{P}_j - C_j \Phi_j \hat{P}_j \\ &= \hat{P}_j - C_j \tilde{P}_{j+1} C_j^T \end{aligned} \tag{I39}$$

Substituting equation (I39) into equation (I37) gives

$$P_j^* = \hat{P}_j - C_j \tilde{P}_{j+1} C_j^T + C_j P_{j+1}^* C_j^T \tag{I40}$$

Rearranging terms then gives

$$P_j^* = \hat{P}_j + C_j (P_{j+1}^* - \tilde{P}_{j+1}) C_j^T \tag{I41}$$

which is the same as equation (I23) above.

Also the definitions in equations (I25) and (I26) give

$$z_j^* = T_{21}^* \tilde{z}_w(j) + T_{22}^* z_{j+1}^* \tag{I42}$$

Substituting for T_{21}^* and T_{22}^* from equations (I30) and (I34) give

$$z_j^* = -R_j^* \Phi_j^{-1} G \tilde{R}_w^{-1}(j) \tilde{z}_w(j) + R_j^* C_j R_{j+1}^{*-1} z_{j+1}^* \tag{I43}$$

Pre-multiplying by R_j^{*-1} and substituting $\Delta x^* = R^{*-1} z^*$ gives

$$\Delta x_j^* = -\Phi_j^{-1} G \tilde{R}_w^{-1}(j) \tilde{z}_w(j) + C_j \Delta x_{j+1}^* \tag{I44}$$

In reference 4, section X.A it is also shown that

$$-\Phi_j^{-1} G \tilde{R}_w^{-1}(j) \tilde{z}_w(j) = \hat{\Delta x}_j - C_j \tilde{\Delta x}_{j+1} \tag{I45}$$

Substituting this into equation (I44) then gives

$$\Delta x_j^* = \hat{\Delta x}_j + C_j (\Delta x_{j+1}^* - \tilde{\Delta x}_{j+1}) \tag{I46}$$

which is the same as equation (I22) above.

APPENDIX J

INVERSION OF UPPER TRIANGULAR MATRICES

Let U be a nonsingular upper triangular matrix of dimension $n \times n$ to be inverted. In the MSF/S implementation, the lower triangular matrix $L = U^{-T}$ is actually computed to take maximum advantage of existing array storage space. Let $u_{i,j} \varepsilon U$ and $\ell_{i,j} \varepsilon L$, then L is computed as follows:

Define

$$\ell_{1,1} = 1/u_{1,1} \tag{J1}$$

For each subsequent row j from $j = 2, \ldots, n$ define

$$\ell_{j,j} = 1/u_{j,j} \tag{J2}$$

$$\ell_{j,k} = -\left(\sum_{i=k}^{j-1} \ell_{i,k} u_{i,j}\right) \ell_{j,j} \qquad k = 1 \ldots, j-1 \tag{J3}$$

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